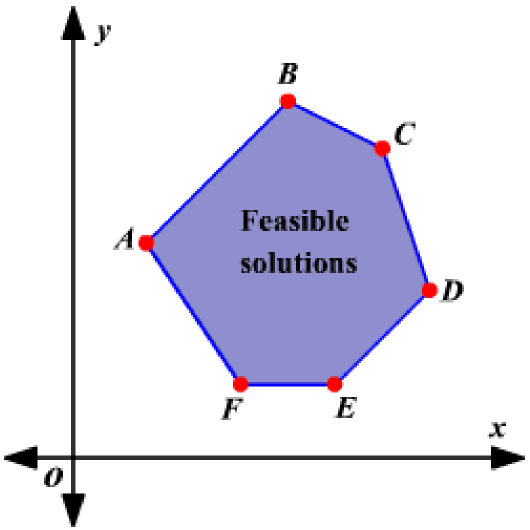
University of Ljubljana, Faculty of Computer and Information Science

Linear programming

for solving problems



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Analysis of Algorithms and Heuristic Problem Solving
Version 2024

Illustration of LP for two variables

The company produces interior and exterior paints from two raw materials, M1 and M2. The table below presents the basic data of the problem

	Tons of raw material per ton of		Maximum daily
	exterior paint	interior paint	available (tons)
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton in €1000	5	4	

- A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

- Determine the optimum product mix of interior and exterior paints that maximizes the daily profit.
- What are decision variables? The objective to optimize? Constraints?

Illustration of LP for two variables

maximize $z = 5x_1 + 4x_2$

with constraints

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_2 \leq 2$

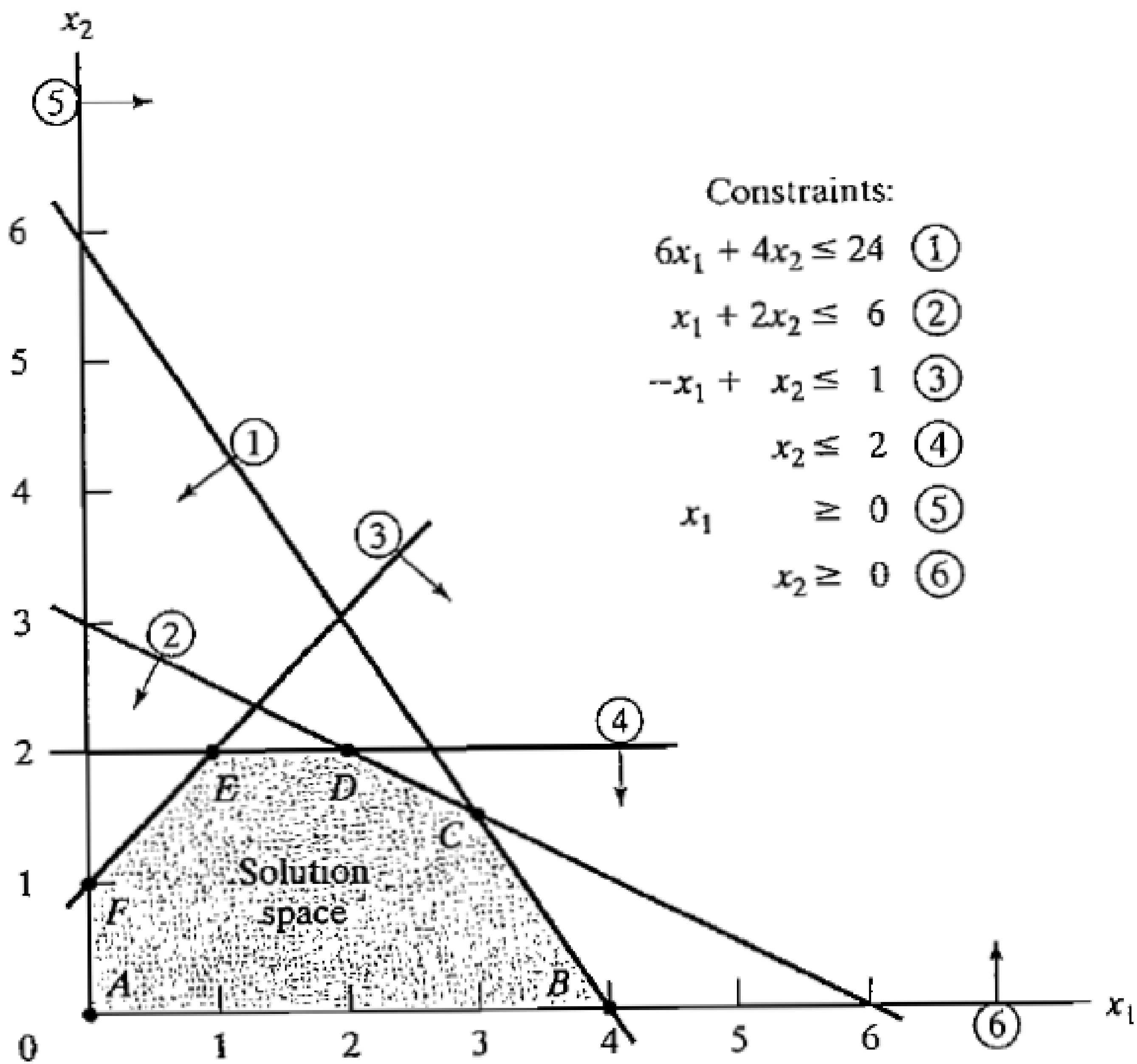
$x1, x2 \geq 0$

check the code in Python

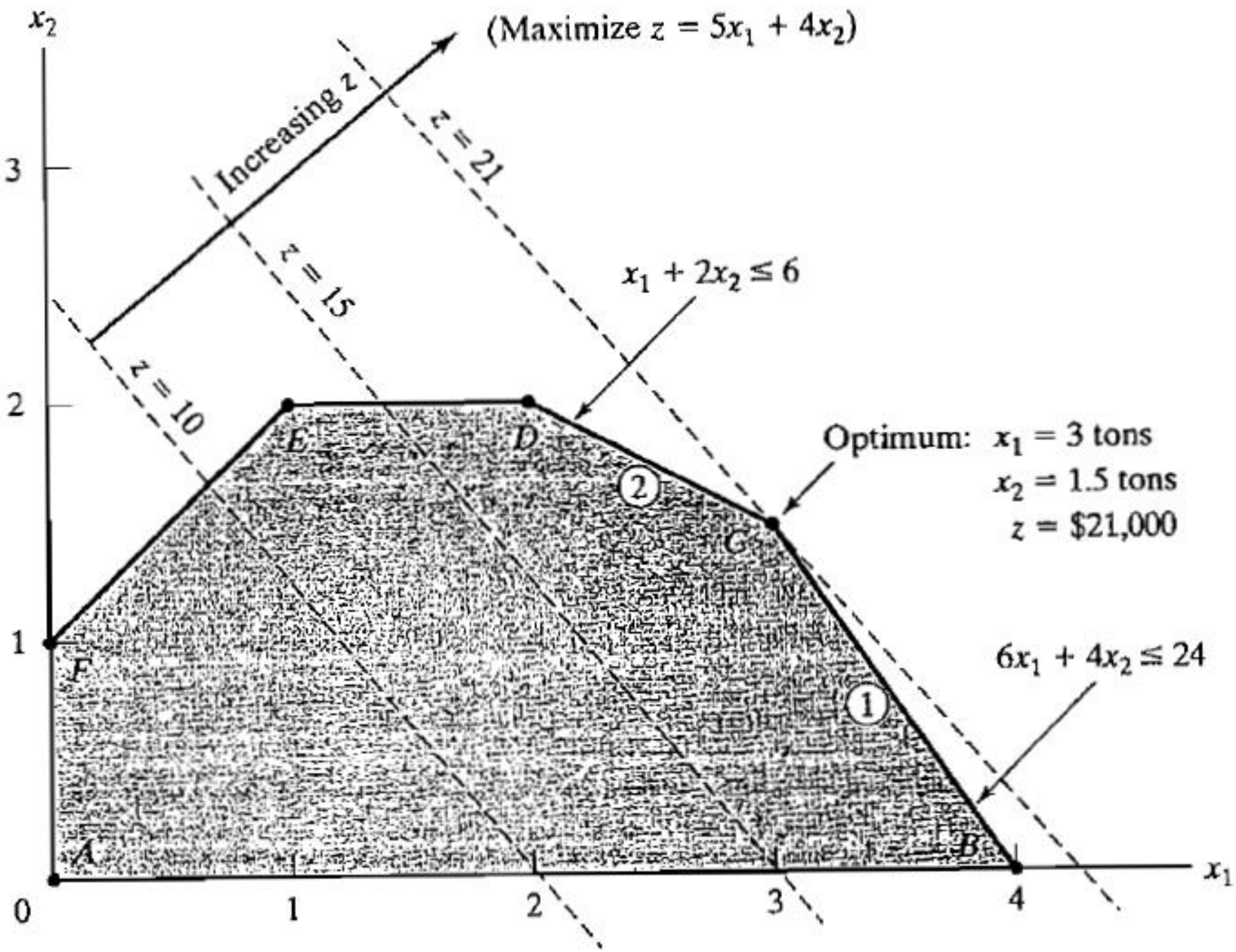
Python code of the LP

```
from scipy.optimize import linprog

# Coefficients of the objective function
c = [-5, -4]  # Note the negative sign because linprog performs minimization

# Coefficients of the inequality constraints (lhs)
A = [
    [6, 4],
    [1, 2],
    [-1, 1],
    [0, 1]
]

# Constants of the inequality constraints (rhs)
b = [24, 6, 1, 2]

# Boundaries for the variables
x0_bounds = (0, None)  # x1 must be >= 0
x1_bounds = (0, None)  # x2 must be >= 0

# Perform linear programming
result = linprog(c, A_ub=A, b_ub=b, bounds=[x0_bounds, x1_bounds], method='highs')

result

# returns x1=3   x2=1.5
#With these values, the maximum value for the objective function z is:
# z=−21  note the negation as the objective function was negated
```

Feasible solutions



Graphical solution



Standard LP problem

- given n real numbers $c_1, c_2, \ldots, c_n$
- m real numbers $b_1, b_2, \ldots, b_m$
- $m \cdot n$ real numbers a_{ij} for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$
- we wish to find n real numbers $x_1, x_2, \ldots, x_n$ that

maximize $\sum_{j=1}^{n} c_j x_j$,

subject to

$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for $i=1,2,\ldots,m$

$x_j \geq 0$ for $j=1,2,\ldots,n$

Matrix notation of LP

maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

Converting LP into the standard form

Four possible transformations

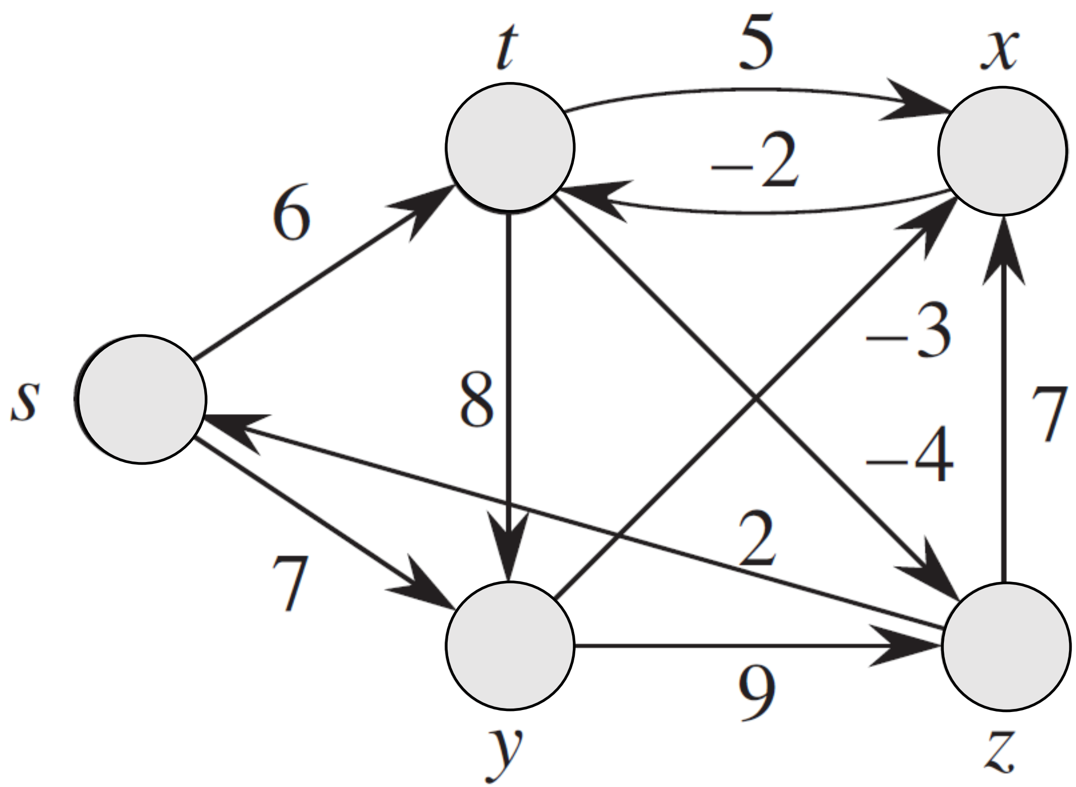
1. the objective function is minimization instead of maximization
2. variables without nonnegativity constraints
3. equality constraints (instead of less-than-or-equal)
4. inequalities in the form of greater-than-or-equal

$$
\begin{array}{lrcrcl}
\text{minimize} & -2x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
& x_1 & & & \geq & 0
\end{array}
$$

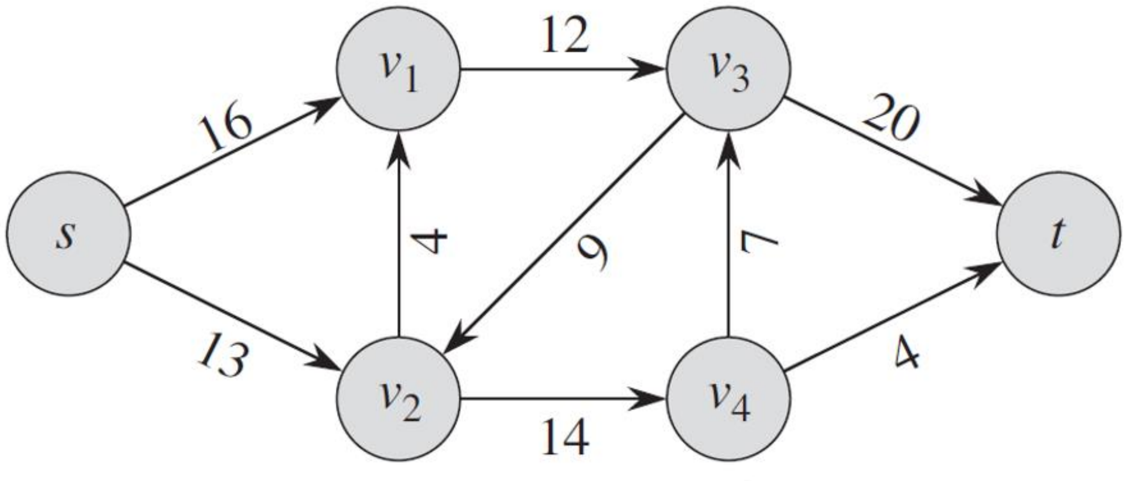
Formulating problems as LPs

- shortest paths
- maximum flow
- minimum-cost flow
- multicommodity flow

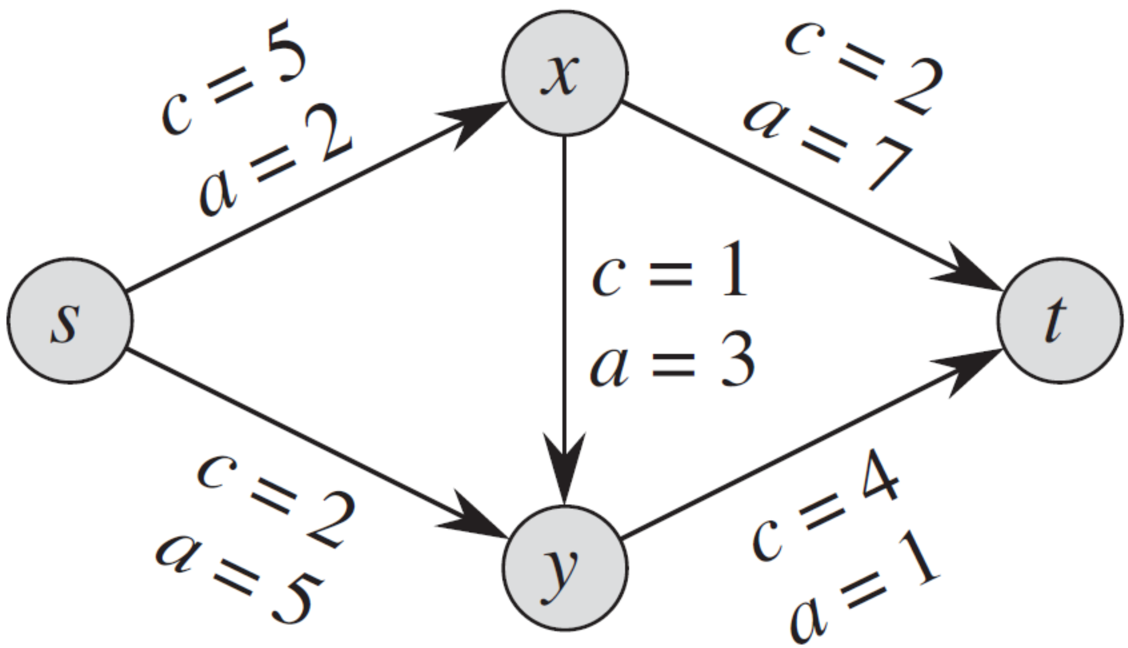
Single-source shortest path problem



Maximum flow problem

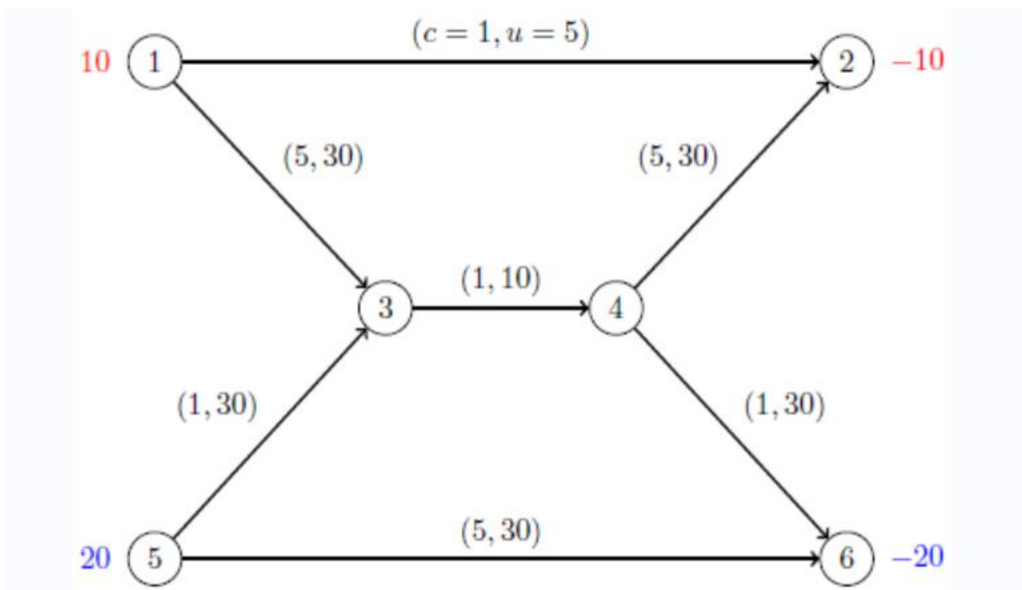


Minimum-cost flow problem



Multicommodity flow

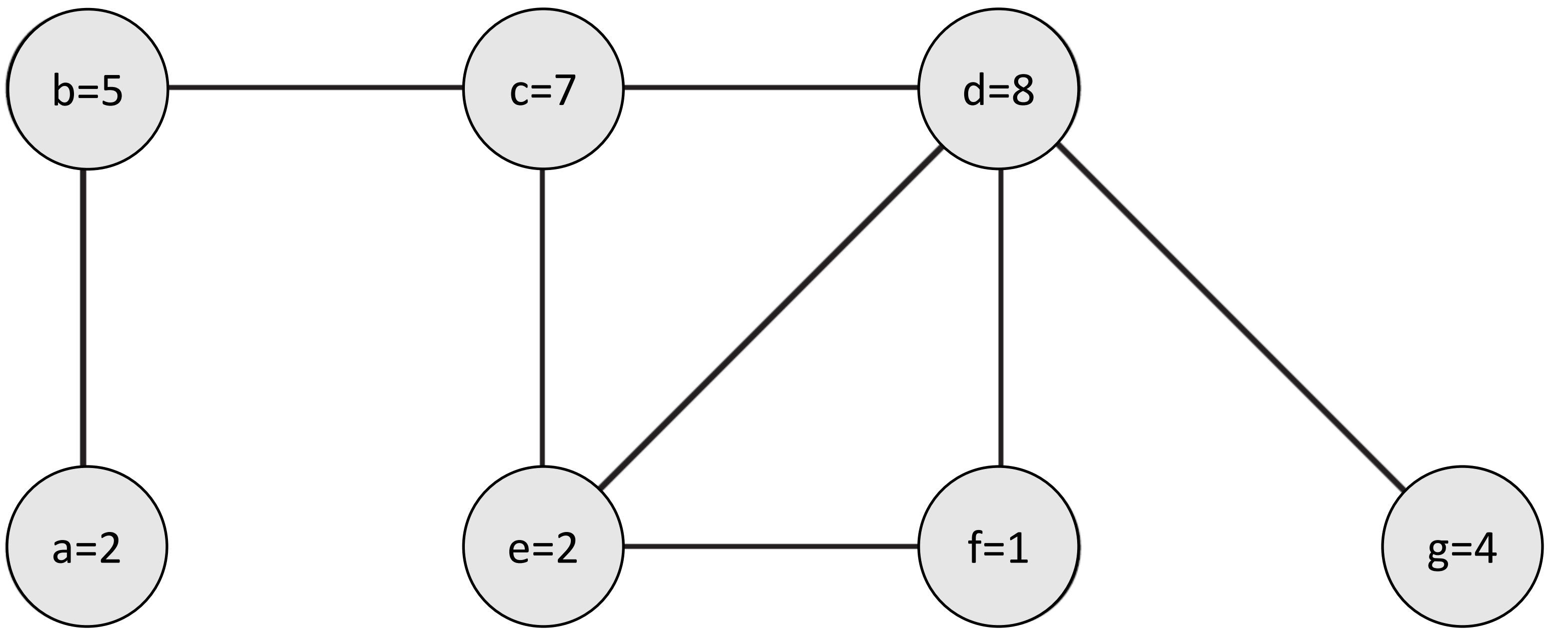
- two commodities with demand 10 and 20
- c=cost, u=capacity



Approximation algorithms and LP

- weighted vertex cover problem
- LP relaxation as approximation technique
- O-1 integer programming

Weighted vertex cover



LP relaxation

APPROX-MIN-WEIGHT-VC(G, w)

$C = \emptyset$
compute $\bar{x}$, an optimal solution to the linear program
for each $v \in V$
    if $\bar{x}(v) \geq 1/2$
        $C = C \cup \{v\}$
return C