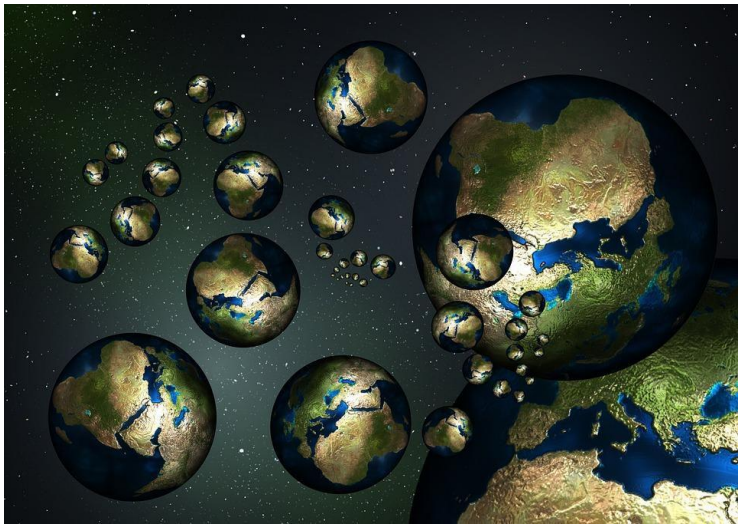
University of Ljubljana, Faculty of Computer and Information Science

Computational complexity of multithreaded algorithms



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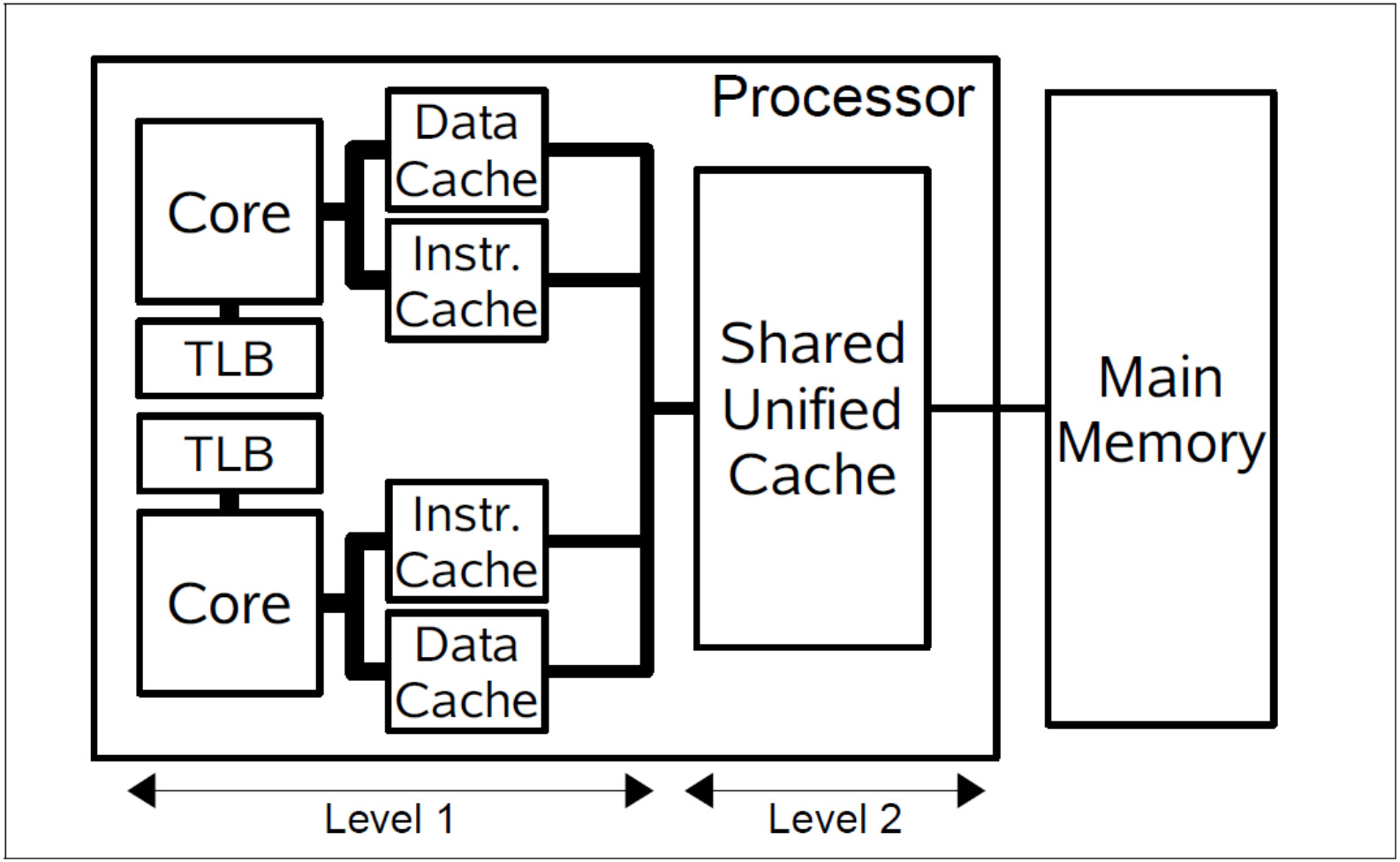
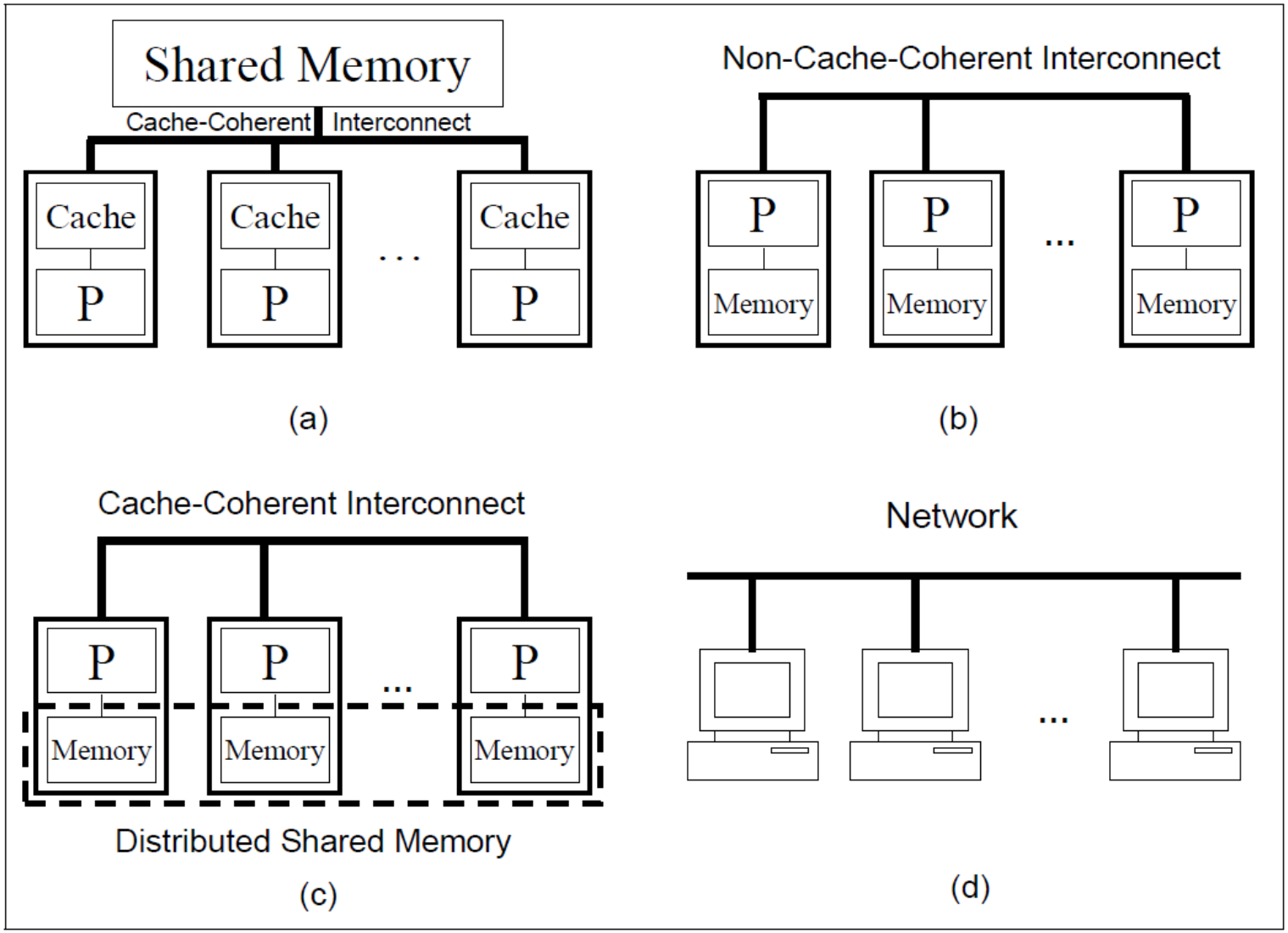
Analysis of Algorithms and Heuristic Problem Solving

Edition 2024

Architecture of multi-core processor



Shared and distributed memory



Problems with parallel execution

- deadlock: each member of a group is waiting for another member, including itself, to take action
- happens under the following Coffman conditions
  - mutual exclusion: at least one resource is held in a non-shareable mode (e.g., entering a critical section)
  - hold and wait (resource holding): a process is holding at least one resource and requesting additional resources which are being held by other processes.
  - no preemption: a resource can be released only voluntarily by the process holding it.
  - circular wait: each process must be waiting for a resource which is being held by another process, which in turn is waiting for the first process to release the resource
- livelock: two or more processes continually repeat the same interaction in response to changes in the other processes without doing any useful work.
- starvation: some resource may always be allocated to some process
- race conditions and synchronization: system attempts to perform two or more operations at the same time, but the operations must be done in the proper sequence to be done correctly.

Race conditions

- deterministic and nondeterministic multithreaded programs

```
void Race() {
   int x = 0 ;
   parallel for i=1 to 2
      x = x +1 ;
   print x ;
}
```

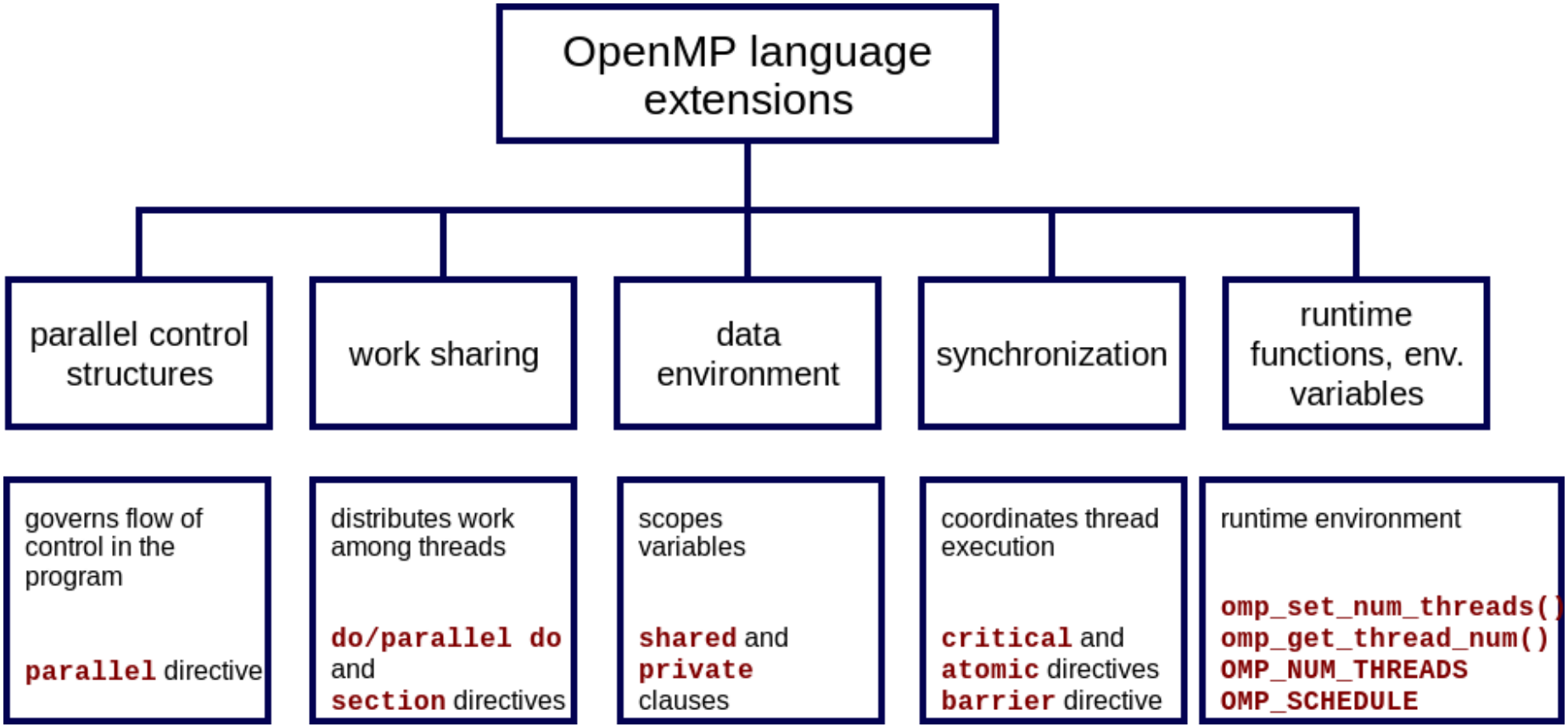
Low level synchronization mechanisms

- monitor: a mechanism that allows threads to have both mutual exclusion and the ability to wait (block) for a certain condition to become false. Monitors also have a mechanism for signaling other threads that their condition has been met.
- semaphore: a (counting) variable controlling access to a common resource
- atomic operations: program operations that cannot be preemptied

Dynamic threads

- simplified programming,
- top-level parallelism
- three new constructs: parallel, spawn, sync
- simplified complexity analysis
- platforms: Cilk, Cilk++, OpenMP, Task Parallel Library (.NET), Threading Building Blocks(C++, Intel), JOMP, JPPF (Java)

OMP elements



Example: Fibonacci numbers

$F_0 = 0$

$F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$ za n >= 2

```
int fib (int n){
  if (n <= 1)
     return n ;
  else {
     int x = fib(n-1) ;
     int y = fib(n-2) ;
     return x + y ;
  }
}
```

- $T(n) = T(n-1) + T(n-2) + \Theta(1)$
- solution $T(n) = \Theta(\tau^n)$
  $\tau = (1 + \sqrt{5})/2$

Multithreaded Fibonacci

```
int pFib (int n){
  if (n <= 1)
     return n ;
  else {
     int x = spawn pFib(n-1) ;
     int y = pFib(n-2) ;
     sync
     return x + y ;
  }
}
```

- nested parallelism
- scheduler

Fibonacci with OpenMP

```
int pFib (int n){
  if (n <= 1)
     return n ;
  else {
      int x, y ;
     # pragma omp sections public(x, y)
     {
        #pragma omp section
        x = pFib(n-1) ;

        #pragma omp section
        y = pFib(n-2) ;
     }
     return x + y ;
  }
}
```

Multithreaded computational model used in computational complexity analysis

- acyclic directed graph
- equal processors
- no resources for scheduling
- total time, time of parallel tasks
- critical path
- number of processors P, time for P processors T_P
- T_1, T_∞

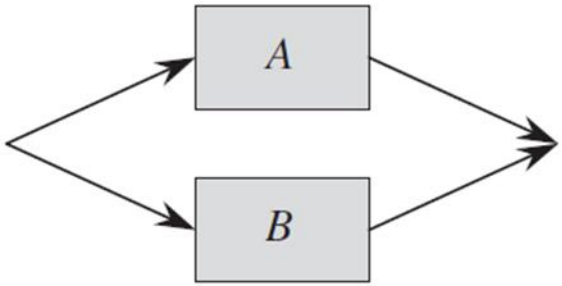
Parallel speedup

- in one step, using P processors, we finish P units of work, in time T_P we do $P \cdot T_P$ units
- total work is T_1, note that $P \cdot T_P \geq T_1$
- speedup rule: $T_P \geq T_1 / P$
- also $T_P \geq T_\infty$
- speedup or level of parallelism is $T_1 / T_P \leq P$
- linear speedup $T_1 / T_P = \Theta(P)$
- ideal linear speedup $T_1 / T_P = P$

Analysis of parallel algorithms

Work: $T_1(A \cup B) = T_1(A) + T_1(B)$

Span: $T_\infty(A \cup B) = T_\infty(A) + T_\infty(B)$



Work: $T_1(A \cup B) = T_1(A) + T_1(B)$

Span: $T_\infty(A \cup B) = \max(T_\infty(A), T_\infty(B))$

Analysis of parallel algorithms

- Fibonacci
- $T_1(n) = T_1(n-1) + T_1(n-2)) + \Theta(1)$

- $T_\infty(n) = \max(T_\infty(n-1) , T_\infty(n-2)) + \Theta(1)$
  $= T_\infty(n-1) + \Theta(1)$
  $= \Theta(n)$

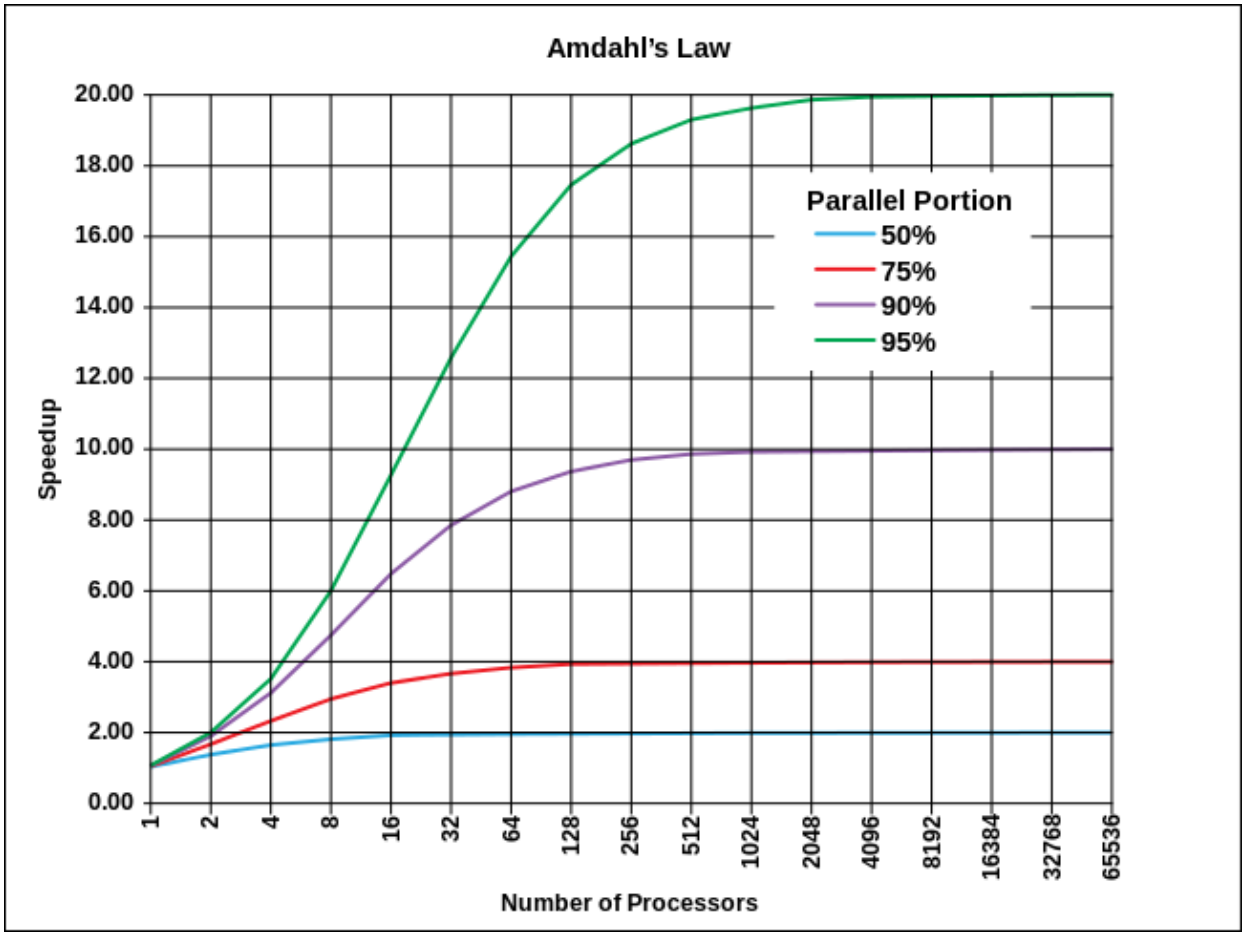
Limits of parallelization

- Amdahl's law

- speedup $S = T_1 / T_P$
- f = proportion of parallelizable code
- $S = \frac{1}{\frac{f}{P} + (1 - f)}$

- let us compute speedup for 2, 5, 10, ∞, processors and f=0.9, 0.5, 0.1

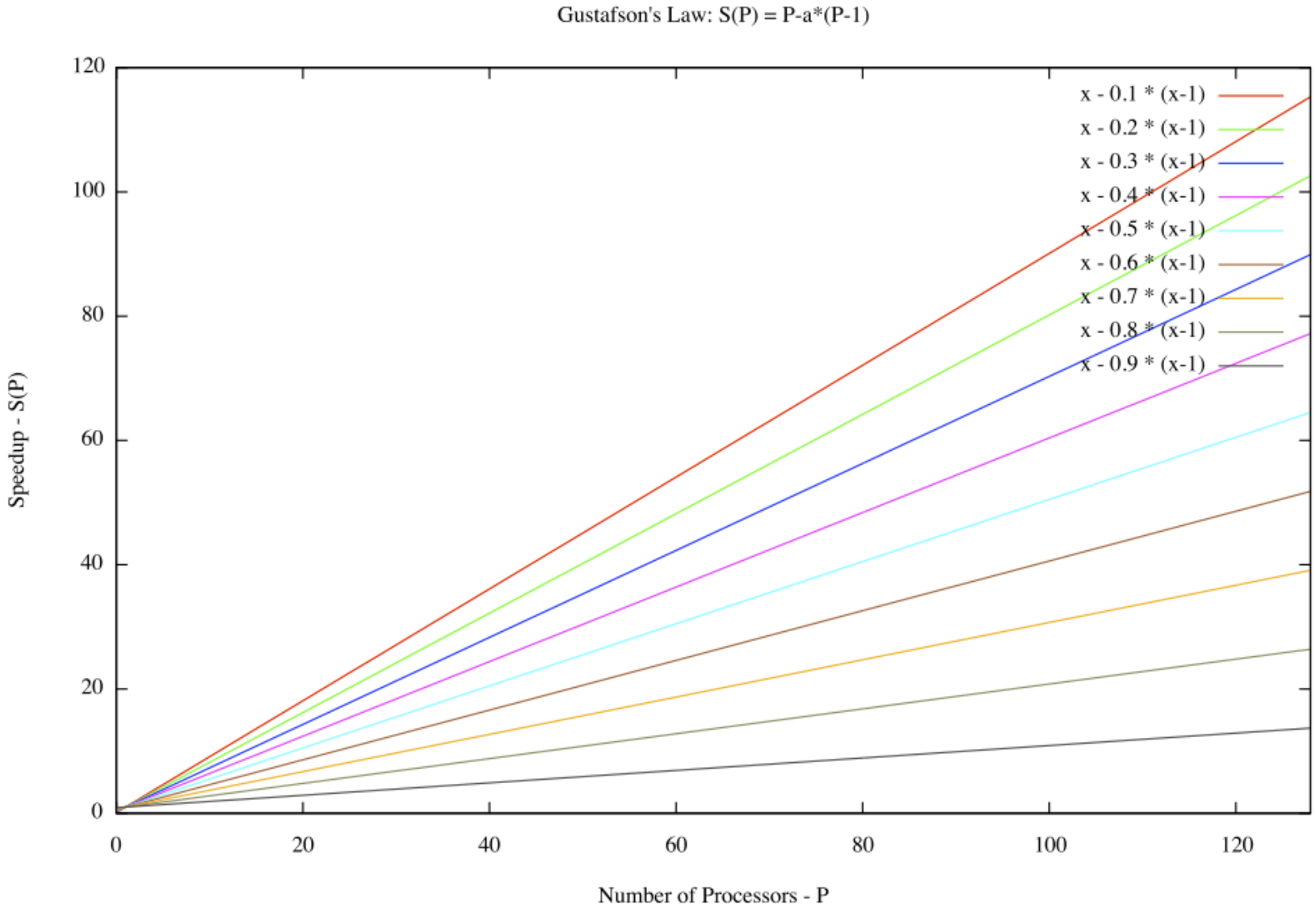
Amdahl's law


Amdahl's Law
Speedup
20.00
18.00
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
Parallel Portion
50%
75%
90%
95%
1
2
4
8
16
32
64
128
256
512
1024
2048
4096
8192
16384
32768
65536
Number of Processors

Gustafson's law

- processing time (on each processor) is split to
  $T_1 = a + b$ (a is sequential time, b = parallel time)
  Sequential share of work $\alpha = a/(a+b)$
  $1 - \alpha$ is a share of parallel work
- assumption: using more parallel units, we can solve larger problems (or more problems in the same time), the size of problems grows linearly with P, therefore $T_1 = a + P \cdot b$
- speedup $S_P = (a + P \cdot b)/(a+b) = \alpha + P(1 - \alpha) = P - \alpha (P - 1)$
- for small α the speedup is almost linear in P

Speedup by Gustafson



Two views of parallel speedup, Amdahl and Gustafson

- Amdahl: if we travel to a destination 100km away and we used 1 hour for one half of the distance, the total average time will never reach 100km/h, no matter how fast we travel the second half
- Gustafson: suppose you travel for some time with a speed lower than 100km/h; if the distance is long enough and there is enough time available, you can still reach arbitrary average speed; e.g., if you travel 1 hour with the speed 50km/h and continue the next hour with 150km/h, the total average speed will be 100km/h (or you can travel next half an hour with 200km/h)

Loop parallelization

- Example: multiplication of a matrix and vector y = A x

```
void mat_vec(matrix A, vector x) {
  int n = A.rows ;
  // let the length of y be n
  for i = 1 to n
    y_i = 0 ;
  for i = 1 to n
    for j = 1 to n
      y_i = y_i + A_ij * x_j ;
  return y ;
}
```

```
void mat_vec(matrix A, vector x) {
  int n = A.rows ;
  // let the length of y be n
  parallel for i = 1 to n
    y_i = 0 ;
  parallel for i = 1 to n
    for j = 1 to n
      y_i = y_i + A_ij * x_j ;
  return y ;
}
```

high-level parallelization

Loop parallelization: actual schedule

```
// the code, a compiler would generate for the main loop
void mat_vec_main_loop(matrix A, vector x, vector y, n, i, k) {
  if (i == k) {
    for j = 1 to n
      y_i = y_i + A_ij * x_j ;
  }
  else {
    mid = (i + k) / 2 ; // the floor
    spawn mat_vec_main_loop(A, x, y, n, i, mid)
    mat_vec_main_loop(A, x, y, n, mid+1, k)
    sync
  }
}
```

The compiler might generate more coarse parallelization