Ime in priimek

Vpisna številka

1	
2	
3	
4	
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Osnove matematične analize: prvi računski izpit

17. januar 2023

Čas pisanja je 90 minut. Dovoljena je uporaba 1 lista A4 formata s formulami. Uporaba kalkulatorja ali drugih pripomočkov ni dovoljena. Vse odgovore dobro utemelji!

1. naloga (25 točk)

a) (10 točk) Poišči vse kompleksne rešitve enačbe $z \cdot \overline{z} + z^2 = 2$.

$z = x + iy$

$$(x+iy)(x-iy) + (x+iy)^2 = 2$$

$$x^2 + y^2 + x^2 + 2ixy - y^2 = 2$$

$$\underbrace{2x^2}_{2} + i\,(\underbrace{2xy}_{0}) = 2$$

$$x = \pm 1 \longrightarrow y = 0$$

$$\underline{\underline{z_{1,2} = \pm 1}}$$

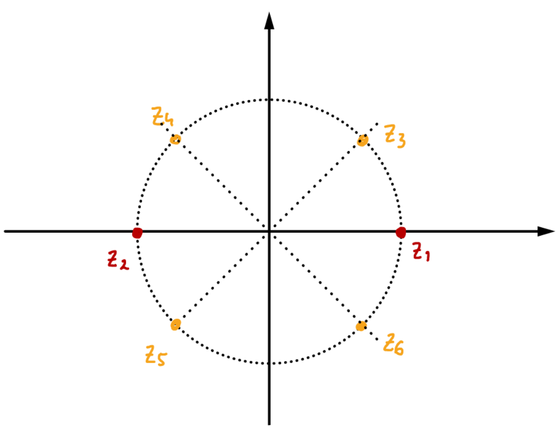
b) (10 točk) Poišči vse kompleksne rešitve enačbe $z^4 = -1$.

$$\left.\begin{aligned} &z = re^{i\varphi}, \quad z^4 = r^4 e^{i\cdot 4\varphi} \\ &-1 = 1 \cdot e^{i\cdot\pi} \end{aligned}\right\} \quad r^4 = 1 \qquad \begin{aligned} 4\varphi &= \pi + 2\pi k \\ \varphi &= \tfrac{\pi}{4} + \tfrac{\pi k}{2}, \quad k = 0,1,2,3 \end{aligned}$$

$$\underline{\underline{z_3 = e^{i\cdot\frac{\pi}{4}}}} \quad \underline{\underline{z_4 = e^{i\cdot\frac{3\pi}{4}}}} \quad \underline{\underline{z_5 = e^{i\cdot\frac{5\pi}{4}}}} \quad \underline{\underline{z_6 = e^{i\cdot\frac{7\pi}{4}}}}$$

c) (5 točk) V kompleksni ravnini nariši vse kompleksne rešitve enačbe

$$(\underbrace{z \cdot \overline{z} + z^2 - 2}_{z_1,\, z_2})(\underbrace{z^4 + 1}_{z_3, \dots, z_6}) = 0.$$



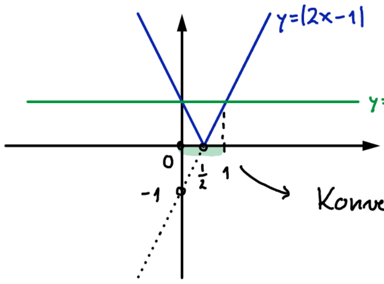
2. naloga (25 točk)

a) (10 točk) Dana je vrsta $\sum_{n=0}^{\infty}(\sqrt{|2x-1|})^{2n}$. Za katere vrednosti $x \in \mathbb{R}$ dana vrsta konvergira?

$$a_n = \left(\sqrt{|2x-1|}\right)^{2n} = \left(\left(\sqrt{|2x-1|}\right)^2\right)^n = |2x-1|^n = q^n \qquad \text{za } q = |2x-1|$$

$\Rightarrow$ to je geometrijska vrsta, ki konvergira za $|q| < 1$

$$|2x-1| < 1$$



Konvergira za $\underbrace{x \in (0,1)}_{0 < x < 1}$.

b) (5 točk) Za vrsto iz b) poišči tak $x \in \mathbb{R}$, da bo vsota vrste enaka 1.

Za $0 < x < 1$ je vsota $\dfrac{1}{1-q}$.

$$\frac{1}{1-q} = 1$$

$$1 = 1-q$$

$$q = 0$$

$$|2x-1| = 0$$

$$2x = 1$$

$$x = \frac{1}{2}$$

c) (10 točk) Z uporabo korenskega kriterija ugotovi, če naslednja vrsta konvergira.

$$\sum_{n=1}^{\infty} \frac{n}{(1+\frac{1}{n})^{n^2}}$$

$$\sqrt[n]{a_n} = a_n^{\frac{1}{n}} = \left(\frac{n}{(1+\frac{1}{n})^{n^2}}\right)^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{(1+\frac{1}{n})^{n^2\cdot\frac{1}{n}}} = \frac{\sqrt[n]{n}}{(1+\frac{1}{n})^n}$$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{(1+\frac{1}{n})^n} = \frac{\lim_{n\to\infty}\sqrt[n]{n}}{\lim_{n\to\infty}(1+\frac{1}{n})^n} = \frac{1}{e} < 1 \Rightarrow \text{vrsta konvergira}$$

3. naloga (25 točk)

Naj bo $f(x,y) = x^3 + 3xy + \frac{3}{2}y^2$.

a) (5 točk) Izračunaj gradient funkcije f v točki $(1,-1)$.

$$\left.\begin{aligned} f_x(x,y) &= 3x^2+3y \\ f_y(x,y) &= 3x+3y \end{aligned}\right\} \quad \begin{aligned} &(\operatorname{grad} f)(x,y) = (3x^2+3y,\ 3x+3y) \\ &(\operatorname{grad} f)(1,-1) = (3-3,\ 3-3) = \underline{\underline{(0,0)}} \end{aligned}$$

b) (10 točk) Določi stacionarne točke funkcije f in jih klasificiraj (če se da).

$$\left.\begin{aligned} f_x = 0 &\longrightarrow 3x^2+3y=0 \longrightarrow x^2+y=0 \longrightarrow y=-x^2 \\ f_y = 0 &\longrightarrow 3x+3y=0 \longrightarrow x+y=0 \longrightarrow y=-x \end{aligned}\right\}$$

$$-x = -x^2$$
$$x = x^2$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$

$x=0$, $y=0$: $\underline{\underline{T_1(0,0)}}$

$x=1$, $y=-1$: $\underline{\underline{T_2(1,-1)}}$

$$f_{xx} = 6x \quad f_{xy} = 3 \quad f_{yy} = 3 \Rightarrow H = \begin{bmatrix} 6x & 3 \\ 3 & 3 \end{bmatrix}$$

- $T_1(0,0)$ $\quad \det H = \begin{vmatrix} 0 & 3 \\ 3 & 3 \end{vmatrix} = -9 < 0 \longrightarrow \underline{\underline{\text{sedlo}}}$
- $T_2(1,-1)$ $\quad \det H = \begin{vmatrix} 6 & 3 \\ 3 & 3 \end{vmatrix} = 18-9 = 9 > 0 \quad f_{xx} > 0 \longrightarrow \underline{\underline{\text{minimum}}}$

c) (10 točk) Določi ekstreme funkcije f na krivulji z enačbo $y = 2x^2$.

$$f(x,y) = x^3 + 3xy + \frac{3}{2}y^2 \qquad y = 2x^2$$

$$\begin{aligned} f(x) &= x^3 + 3\cdot x \cdot 2x^2 + \frac{3}{2}\cdot 4x^4 \\ &= x^3 + 6x^3 + 6x^4 \\ &= 7x^3 + 6x^4 \end{aligned}$$

$$f'(x) = 21x^2 + 24x^3 = 0 \quad /:3$$

$$7x^2 + 8x^3 = 0$$

$$x^2(7+8x) = 0$$

$x = 0$, $y = 2\cdot 0^2 = 0$: $\underline{\underline{T_1(0,0)}}$

$7+8x=0$: $x = -\frac{7}{8}$, $y = 2\left(-\frac{7}{8}\right)^2 = 2\cdot\frac{49}{64} = \frac{49}{32}$: $\underline{\underline{T_2\left(-\frac{7}{8}, \frac{49}{32}\right)}}$

4. naloga (25 točk)

a) (10 točk) Pokaži, da je

$$\int \frac{\log x}{x^2}dx \quad = -\frac{1}{x}(\log x + 1) + c.$$

1. način:

$$\left(-\frac{1}{x}(\log x+1)+c\right)' = \left(-\frac{1}{x}\right)'(\log x+1) + \left(-\frac{1}{x}\right)(\log x+1)' = \frac{1}{x^2}(\log x+1) - \frac{1}{x}\left(\frac{1}{x}\right) = \frac{1}{x^2}\log x + \frac{1}{x^2} - \frac{1}{x^2} =$$

$$= \frac{\log x}{x^2} \checkmark$$

2. način: $u = \log x$ $\quad du = \frac{1}{x}dx$

$dv = x^{-2}dx$ $\quad v = -\frac{1}{x}$

$$\int u\,dv = uv - \int v\,du = -\frac{1}{x}\log x - \int -\frac{1}{x}\cdot\frac{1}{x}dx = -\frac{1}{x}\log x + \int x^{-2}dx = -\frac{1}{x}\log x - \frac{1}{x} + c =$$

$$= -\frac{1}{x}(\log x + 1) + c$$

b) (5 točk) Izračunaj $\lim_{x\to\infty}\left(-\frac{1}{x}(\log x + 1)\right)$.

$$\lim_{x\to\infty}\frac{\overset{\to\infty}{\log x+1}}{\underset{\to -\infty}{-x}} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{-1} = -\lim_{x\to\infty}\frac{1}{x} = \underline{\underline{0}}$$

c) (10 točk) Izračunaj volumen vrtenine, ki jo dobimo, če graf funkcije

$$f(x) = \frac{\sqrt{\log x}}{x}$$

zavrtimo okrog x-osi na intervalu $[1, \infty]$.

$$V = \int_1^\infty \pi f(x)^2\,dx = \int_1^\infty \pi\cdot\frac{(\sqrt{\log x})^2}{x^2}dx = \pi\int_1^\infty\frac{\log x}{x^2}dx = \pi\left(-\frac{1}{x}(\log x+1)\right)\Big|_1^\infty =$$

$$= \pi\left(\underbrace{\lim_{x\to\infty}\left(-\frac{1}{x}(\log x+1)\right)}_{0} - \left(-\frac{1}{1}\right)(\underbrace{\log 1}_{0}+1)\right) = \pi(0+1) = \underline{\underline{\pi}}$$