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TOPOLOGICAL DATA ANALISYS

Solved problems

Ljubljana, 2022

Topological Data Analysis
Solved problems
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Preface

This collection of problems has not been checked by a third party and is sure to contain some mistakes. If you think you have found some or if you have any questions, feel free to stop by during the office hours or contact me by email.

Shortcuts have been added to help you navigate the document. Clicking on the symbol ⇩ next to a problem will take you to the solution. Clicking on the symbol ⇧ next to the solution will take you back to the problem. I recommend you spend a bit of time trying to solve each problem on your own before reading the solution. Check the solution even if you solve the problem successfully as it might contain a bit of additional information and suggestions for further explorations.

Notation

$\mathbb{N}$	...	the set of positive integers (natural numbers)
$\mathbb{Z}$	...	the set of integers
$\mathbb{R}$	...	the set of real numbers
X, Y	...	topological spaces
$\mathcal{P}(X)$	...	the power set of X
$\mathcal{T}$	...	topology on X
$\mathbb{R}^n$	...	the n-dimensional Euclidean space
$\\|x\\|_p$	...	the p-norm of $x \in \mathbb{R}^n$
$d_p(\cdot\,,\cdot)$	...	the p-metric on $\mathbb{R}^n$
$B(x_0, r)$	...	open ball with centre at x_0 and radius r
$\overline{B}(x_0, r)$	...	closed ball with centre at x_0 and radius r
S^n	...	the n-dimensional sphere
B^n	...	the n-dimensional ball
T	...	the torus $S^1 \times S^1$
P	...	the real projective plane $\mathbb{R}P^2$
T	...	the Klein bottle
$X \cong Y$	...	X and Y are homeomorphic
$X \simeq Y$	...	X and Y are homotopy equivalent
$\overrightarrow{a} \times \overrightarrow{b}$	...	cross product of $\overrightarrow{a}$ and $\overrightarrow{b}$
$\overrightarrow{a} \cdot \overrightarrow{b}$	...	scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$
id_X	...	the identity function on X
S	...	a point cloud $S \subset \mathbb{R}^n$
$\mathrm{Conv}(S)$	...	the convex hull of S
V_s	...	the Voronoi region of $s \in S$
$\mathcal{D}(S)$	...	the Delaunay triangulation of S
σ	...	a simplex
$\tau < \sigma$	...	τ is a face of σ
$\sigma > \tau$	...	σ is a co-face of τ
$\mathrm{cl}(\sigma)$	...	closure of σ
$\mathrm{st}(\sigma)$	...	open star of σ
$\mathrm{lk}(\sigma)$	...	link of σ
$\chi(X)$	...	the Euler characteristic of X
$X \# Y$	...	connected sum of X and Y
$\mathrm{Rips}(S, r)$	...	Vietoris-Rips complex of a set S
$\mathrm{Cech}(S, r)$	...	Čech complex of a set S
K_n	...	the complete graph on n vertices
$\mathrm{im} f$	...	the image (range) of map f
$\ker f$	...	the kernel (nullspace) of map f
$W_\infty(\cdot\,,\cdot)$	...	the bottleneck distance between persistence diagrams
$W_q(\cdot\,,\cdot)$	...	the Wasserstein distance between persistence diagrams for $q \in \mathbb{R}$
$\alpha^{(p)}$	...	a p-dimensional simplex α

Contents

CHAPTER 1

Metric spaces and homeomorphisms

A **topology** on X is a family $\mathcal{T}$ of subsets of X, $\mathcal{T} \subseteq \mathcal{P}(X)$, such that

i. $\emptyset \in \mathcal{T}$,
ii. $X \in \mathcal{T}$,
iii. if I is an arbitrary index set and $U_i \in \mathcal{T}$ for all $i \in I$, then the union $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$ and
iv. if J is a finite index set and $U_j \in \mathcal{T}$ for all $j \in J$, then the intersection $\bigcap_{j \in J} U_j$ is in $\mathcal{T}$.

Elements of $\mathcal{T}$ are called **open sets**.

PROBLEM 1.
Find all topologies on

a. $X = \{0, 1\}$,
b. $Y = \{0, 1, 2\}$.

A **metric** on X is a function $d\colon X \times X \to [0, \infty)$, such that for all $x, y, z \in X$

i. $d(x, y) \geq 0$, (non-negativity)
ii. $d(x, y) = 0$ if and only if $x = y$, (identity of indiscernibles)
iii. $d(x, y) = d(y, x)$ and (symmetry)
iv. $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

An **open ball** with centre $x_0 \in X$ and radius $r \geq 0$ is defined as

$$B(x_0, r) = \{x \in X \;;\; d(x_0, x) < r\}.$$

A **closed ball** with centre $x_0 \in X$ and radius $r \geq 0$ is defined as

$$\overline{B}(x_0, r) = \{x \in X \;;\; d(x_0, x) \leq r\}.$$

PROBLEM 2.
Define $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$d(x, y) = \begin{cases} 0, & x = y, \\ |x| + |y|, & x \neq y. \end{cases}$$

Describe and draw $B(1, 2)$, $B(2, 3)$, $B(-1, 3)$ and $B(2, 1)$.

For $p \in [1, \infty)$ the **p-norm** on $\mathbb{R}^n$ is defined as

$$\|(x_1, \ldots, x_n)\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

For vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ the **p-metric** on $\mathbb{R}^n$ is defined by

$$d_p(x, y) = \|x - y\|_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}.$$

Additionally, we define

$$\|(x_1, \ldots, x_n)\|_\infty = \max_i |x_i|$$

and

$$d_\infty(x, y) = \|x - y\|_\infty = \max_i |x_i - y_i|.$$

Problem 3.
Draw $B(0, 1)$ and $\overline{B}(0, 1)$ in $\mathbb{R}^2$ for $p \in \{1, 2, \infty\}$.

Problem 4.
Define $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$d((x_1, x_2), (y_1, y_2)) = \begin{cases} \|(x_1, x_2) - (y_1, y_2)\|_2, & (0, 0), (x_1, x_2), (y_1, y_2) \text{ are collinear,} \\ \|(x_1, x_2)\|_2 + \|(y_1, y_2)\|_2, & \text{otherwise.} \end{cases}$$

Draw the open balls

a. $B((0, 0), 1)$,
b. $B((3, 0), 4)$ and
c. $B((1, 1), \sqrt{2})$.

A **path** on X is a (continuous) map $\gamma\colon [0, 1] \to X$.

Problem 5.
Given the points $A(3, -4)$ and $B(4, 3)$ in $\mathbb{R}^2$ find the parametrization for least three different paths

$$\alpha, \beta, \gamma\colon [0, 1] \to \mathbb{R}^2$$

from A to B.

Problem 6.
Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Find an infinite number of different paths in S^1 from the point $(1, 0)$ to the point $(0, 1)$.

A **homeomorphism** between topological spaces X and Y is a map $f\colon X \to Y$ such that

i. f is a bijection (one-to-one and surjective),
ii. f is continuous and
iii. the inverse f^{-1} is continuous.

If such a map exists, then spaces X and Y are said to be **homeomorphic**.

Let X and Y be topological spaces. If we can find continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, then X and Y are homeomorphic.

To prove that two spaces are **not** homeomorphic, we need to find a topological invariant that distinguishes between the two. The following properties are topological invariants:

- dimension,
- path-connectedness,
- number of connected components,
- Euler characteristic χ,
- orientability,
- existence of non-trivial loops,
- fixed point property,
- ...

PROBLEM 7.
Let $X = (-1, 1)$ and $Y = \mathbb{R}$. Show that the maps

$$f\colon (-1,1) \to \mathbb{R}, \quad f(x) = \frac{x}{\sqrt{1-x^2}}$$

and

$$g\colon \mathbb{R} \to (-1,1), \quad g(x) = \frac{x}{\sqrt{1+x^2}}$$

are homeomorphisms. Can you find another pair of maps that shows X and Y are homeomorphic? Can you show that $(-1, 1)$ and $[-1, 1]$ are not homeomorphic?

PROBLEM 8.
Let $n \geq 1$, $X = S^n \setminus \{(0, \ldots, 0, 1)\} \subseteq \mathbb{R}^{n+1}$ and $Y = \mathbb{R}^n$. Show that the maps $f\colon X \to Y$ and $g\colon Y \to X$, defined by

$$f(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n)$$

and

$$g(x_1, \ldots, x_n) = \left(\frac{2x_1}{x_1^2 + \ldots + x_n^2 + 1}, \ldots, \frac{2x_n}{x_1^2 + \ldots + x_n^2 + 1}, \frac{x_1^2 + \ldots + x_n^2 - 1}{x_1^2 + \ldots + x_n^2 + 1}\right),$$

are homeomorphisms.

PROBLEM 9.
Let $X = S^1 \times [0, 1] \subset \mathbb{R}^3$ and $Y = \{(x, y) \in \mathbb{R}^2 \ ; \ 1 \leq x^2 + y^2 \leq 4\}$. Prove that X and Y are homeomorphic.

PROBLEM 10.
Let $X = \{(x, y, z) \in \mathbb{R}^3 \ ; \ z^2 = x^2 + y^2, \ 0 < z < 1\}$ and $Y = S^1 \times (0, 1)$. Show that $X \cong Y$.

PROBLEM 11.
Let $X_n = S^n \setminus \{(0, \ldots, 0, 1), (0, \ldots, 0, -1)\} \subset \mathbb{R}^{n+1}$ and $Y_n = S^{n-1} \times (-1, 1) \subset \mathbb{R}^{n+1}$. Prove that X and Y are homeomorphic.

PROBLEM 12.
Let $X = [1, 2] \cup (3, 4]$ and $Y = [1, 3]$. Find a map $f\colon X \to Y$ such that f is continuous and a bijection. Show that f^{-1} is not continuous. Why are the X and Y not homeomorphic?

A **homotopy** between two continuous maps $f, g\colon X \to Y$ is a continuous function $H\colon X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

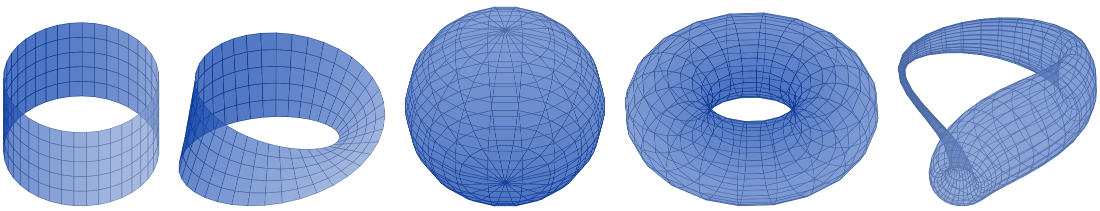
Topological spaces X and Y are **homotopy equivalent** if there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

To prove that two spaces are **not** homotopy equivalent, we need to find a homotopy invariant that distinguishes between the two. The following properties are homotopy invariant:

- path-connectedness,
- number of connected components,
- Euler characteristic χ,
- existence of non-trivial loops,
- homology groups,
- contractibility,
- ...

Any homotopy invariant is also a topological property. The converse is not true. For example, orientability and dimension are topological properties that are not homotopy invariants.

PROBLEM 13.
Which of the following surfaces (cylinder, Moebius strip, sphere, torus, Klein bottle) are homeomorphic? Are any of them homotopy equivalent? If so, which? If not, why not?



PROBLEM 14.
Show that $X = S^n$ and $Y = \mathbb{R}^{n+1} \setminus \{0^{n+1}\}$ are homotopy equivalent.

PROBLEM 15.
Show that the Moebius band $M = [-1,1] \times [-1,1]/_\sim$, where $(-1, y) \sim (1, -y)$ for all $y \in [-1,1]$, is homotopy equivalent to the circle $S^1 = [-1,1]/_\sim$, where $-1 \sim 1$.

PROBLEM 16.
Show that $X = [-1,1] \times [0,1]$ is homotopy equivalent to

$$Y = [-1,1] \times \{0\} \cup \{-1,1\} \times [0,1].$$

A space is **contractible** if it is homotopy equivalent to a point.

PROBLEM 17.
Show that the standard n-simplex Δ^n is contractible.

PROBLEM 18.
The cone of X is the quotient $CX = (X \times [0,1])/_\sim$, where $(x,1) \sim (y,1)$ for all $x, y \in X$. Show that CX is contractible.

CHAPTER 2

Geometry

A line p given by a point $A \in p$ and a direction vector $\vec{p}$ can be parametrized as
$$p\colon \vec{r}_A + t\vec{p}, \quad t \in \mathbb{R}.$$

In cartesian coordinates, a line through the points $A(x_A, y_A)$ and $B(x_B, y_b)$ is given by
$$y = \frac{y_B - y_A}{x_B - x_A}(x - x_A) + y_A.$$

The area of the triangle spanned by the vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ is
$$S = \frac{1}{2}\|\vec{a} \times \vec{b}\|.$$

For any vectors $\vec{a}$ and $\vec{b}$ we have
$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\varphi.$$

Triangle inequalities. In a triangle ΔABC we have
$$\|AB\| \leq \|AC\| + \|CB\|, \quad \|BC\| \leq \|BA\| + \|AC\| \quad \text{and} \quad \|AC\| \leq \|AB\| + \|BC\|.$$
One of the inequalities is an equality if and only if A, B and C lie on the same line.

Heron's formula. The area of a triangle ABC with the semiperimeter
$$s = \frac{1}{2}(\|AB\| + \|BC\| + \|CA\|)$$
is
$$\sqrt{(s - \|AB\|)(s - \|BC\|)(s - \|CA\|)}.$$

Problem 19.

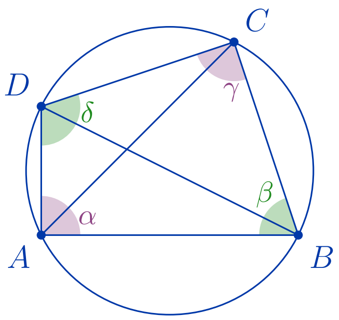
Given three points $A, B, C \in \mathbb{R}^n$, determine whether they are collinear. Do so in at least six different ways:

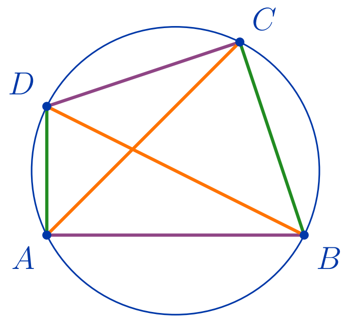
a. calculate the area of the triangle ΔABC using the cross product formula,
b. use Heron's formula to find the area,
c. use the scalar product equality to find one of the angles,
d. try writing $\overrightarrow{AC}$ as a multiple of $\overrightarrow{AB}$,
e. parametrize the line AB and determine if C lies on it,
f. use the triangle inequalities.

Which of these methods seem most efficient? What are the drawbacks for others? Which of these methods can be used to determine the order of the points on the line if they are collinear? How would you design the test cases to determine which of these methods works best.

A quadrilateral $ABCD$ is cyclic if and only if the opposite angles are supplementary:

$$\alpha + \gamma = \beta + \delta = \pi.$$





Ptolemy's theorem. A quadrilateral $ABCD$ is cyclic if and only if

$$\|AB\|\|CD\| + \|BC\|\|AD\| = \|AC\|\|BD\|.$$

Brahmagupta's formula. The area of a cyclic quadrilateral with the semiperimeter

$$s = \frac{1}{2}(\|AB\| + \|BC\| + \|CD\| + \|AD\|)$$

is

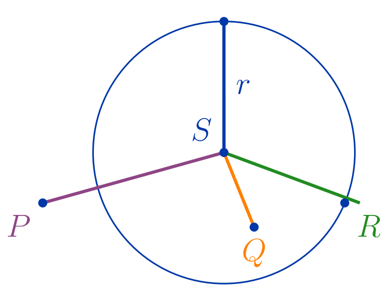
$$S = \sqrt{(s - \|AB\|)(s - \|BC\|)(s - \|CD\|)(s - \|AD\|)}.$$

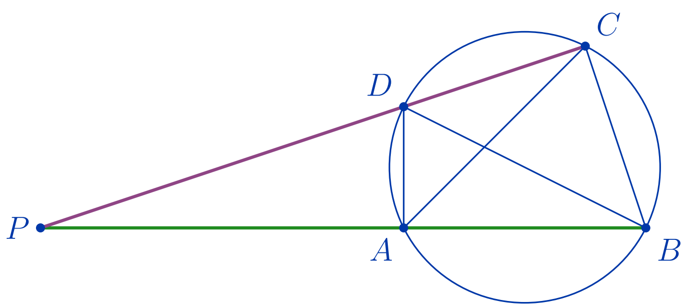
Power-of-a-point. Given a point P, a circle $\mathcal{C}$ with centre S and radius r, and a line through P intersecting the circle at A and B, we have

$$h(P, \mathcal{C}) = \|PS\| - r^2 = \|PA\|\|PB\|.$$

Obviously,

a. $h(P, \mathcal{C}) > 0$ for points P that lie outside of the circle,
b. $h(Q, \mathcal{C}) < 0$ for points Q that lie inside the circle and
c. $h(R, \mathcal{C}) = 0$ for points R that are on the circle.





Intersecting chords theorem. Given four points A, B, C and D, let P denote the intersection of lines AB and CD. Then A, B, C and D are cocyclic if and only if

$$\|PA\|\|PB\| = \|PC\|\|PD\|.$$

Problem 20.

Given four points, A, B, C and D in $\mathbb{R}^2$, determine whether they are cocyclic. Do so in at least six different ways:

a. calculate the opposite angles,
b. use the incircle test,
c. use Ptolemy's theorem,
d. use the Power-of-a-Point theorem,
e. find the equation of the circumcircle of the triangle ABC and check if it holds for the point D,
f. use the Brahmagupta's formula.

Which of these methods seems most efficient? What are the drawbacks for others? Can any of these be used to determine if the fourth point lies inside or outside of the circumcircle through the other three? How does the problem change if the given points are in $\mathbb{R}^n$, $n > 2$. How would you design the test cases to determine which of these methods works best?

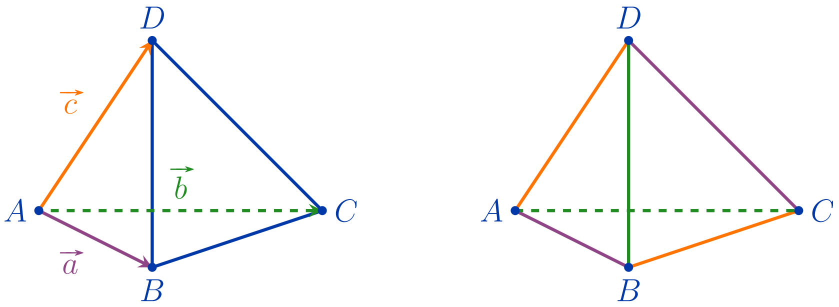
Problem 21.
Given a triangle ΔABC in $\mathbb{R}^2$, find the coordinates of its circumcentre in at least three different ways:

a. find a point S equidistant from A, B and C,
b. write the parametrized (vector) equations for the segment bisectors and find their intersection,
c. write the equations for the segment bisectors in cartesian coordinates and find their intersection.

Which of these methods seems most efficient?

The volume of the tetrahedron spanned by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ is

$$V = \frac{1}{6}\|\vec{a}\cdot(\vec{b}\times\vec{c})\|.$$



Heron-type formula for the volume of a tetrahedron. Define

$$\begin{aligned}
X &= (\|BD\|+\|AD\|-\|AB\|)(\|BD\|+\|AD\|+\|AB\|),\\
Y &= (\|CD\|+\|AD\|-\|BC\|)(\|CD\|+\|AD\|+\|BC\|),\\
Z &= (\|AD\|+\|CD\|-\|AC\|)(\|AD\|+\|CD\|+\|AC\|),\\
x &= (\|AB\|-\|AD\|+\|BD\|)(\|AB\|+\|AD\|-\|BD\|),\\
y &= (\|BC\|-\|BD\|+\|CD\|)(\|BC\|+\|BD\|-\|CD\|),\\
z &= (\|AC\|-\|CD\|+\|AD\|)(\|AC\|+\|CD\|-\|AD\|)
\end{aligned}$$

and

$$a=\sqrt{xYZ},\ b=\sqrt{yZX},\ c=\sqrt{zXY},\ d=\sqrt{xyz}.$$

Then the volume of the tetrahedron $ABCD$ is

$$V = \frac{\sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}}{192\|CD\|\|AD\|\|BD\|}.$$

PROBLEM 22.

Given four points, A, B, C and D in $\mathbb{R}^n$ ($n \geq 3$), determine if they are coplanar in at least four different ways:

a. calculate the volume of the tetrahedron $ABCD$,
b. check if $\overrightarrow{AD}$ can be written as the linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$,
c. find the equation of the plane containing A, B and C and check if it also holds for D,
d. check if the lines AB and CD intersect or are parallel.

Which of these methods seems most efficient?

CHAPTER 3

Triangulations and simplicial complexes

A **triangulation** of a polygonal region $D \subset \mathbb{R}^2$ is a decomposition of D into triangles, such that

- no triangle is degenerate,
- interiors of the triangles are disjoined and
- the intersection of any two triangles is a common edge, a common vertex or empty.

A **triangulation** on a finite set of points $S \subset \mathbb{R}^2$ is any triangulation of its convex hull.

The **convex hull** of a set $S \subset \mathbb{R}^2$, $\mathrm{Conv}(S)$, is the smallest convex set containing S.

Line sweep triangulation. Assume no two points of a finite set $S \subset \mathbb{R}^2$ have the same first coordinate. Then

- find a vertical line such that all points lie to its right,
- let the line sweep horizontally towards the right,
- every time the line meets a point of S add all those edges connecting the new point to the points to its left that can be added without creating intersections (giving priority to the points that were met earlier) and
- every time three edges form a triangle, add its interior.

Once all the points of S lie to the left of the sweeping line the triangulation is complete. The angle of the line and the angle of the sweep can be changed arbitrarily as long as they remain distinct. The requirement that the line never hits more than one point at the time can be relaxed if a means of breaking ties is given (in the case of a vertical line any points that lie on the same vertical line could be ordered by height, for example).

Let $S \subset \mathbb{R}^2$ be a finite set of points such that no four points lie on the same circle. The **Voronoi region** of $s \in S$ is

$$V_s = \{x \in \mathbb{R}^2 \; ; \; d(x,s) \leq d(x,u) \text{ for all } u \in S \setminus \{s\}\}.$$

Non-empty intersections of Voronoi regions are (bounded or unbounded) **Voronoi edges** that lie on the bisectors of the edges between the points of S and **Voronoi vertices** which coincide with some of the circumcenters of the triangles with vertices in S.
The **Voronoi decomposition** of S is the collection of Voronoi regions, Voronoi edges and Voronoi vertices of S.

The **Delaunay triangulation** on S, $\mathcal{D}(S)$ is a (unique) triangulation of S, such that

- the vertices of $\mathcal{D}(S)$ are all points of S,
- xy is an edge of $\mathcal{D}(S)$ if and only if $V_x \cap V_y \neq \emptyset$ and
- xyz is a triangle of $\mathcal{D}(S)$ if and only if $V_x \cap V_y \cap V_z \neq \emptyset$.

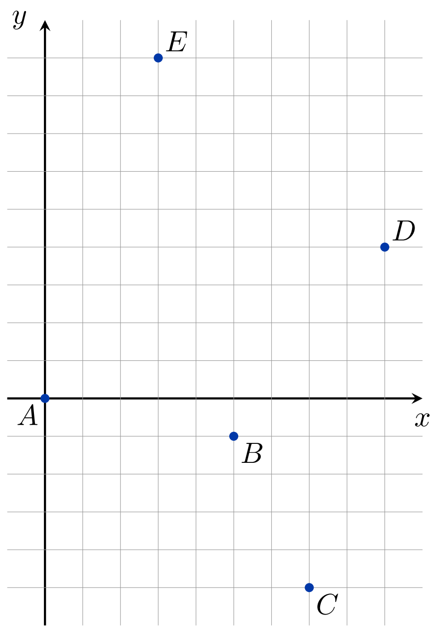
To construct the Delaunay triangulation on S, take any triangulation $\mathcal{T}$ of S and continue flipping edges as long as possible: If ABC and ACD are two triangles of $\mathcal{T}$ and

$$\sphericalangle ABC + \sphericalangle BCD > \pi,$$

then replace them by the triangles ABD and BCD (i.e. replace the edge AC by the edge BD).
An alternative (more geometric) approach: If the point D lies inside the circumcircle of the triangle ABC, replace the triangles ABC and ACD by the triangles ABD and BCD.

Problem 23.
Let $S = \{A(0,0), B(5,-1), C(7,-5), D(9,4), E(3,9)\} \subset \mathbb{R}^2$.

a. Construct the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of S using vertical line sweep from left to right and the horizontal line sweep upwards.
b. We can get the Delaunay triangulation on S by flipping certain edges. How many edge flips are necessary to produce a Delaunay triangulation from $\mathcal{T}_1$? From $\mathcal{T}_2$?
c. Draw the corresponding Voronoi diagram. Is it unique?

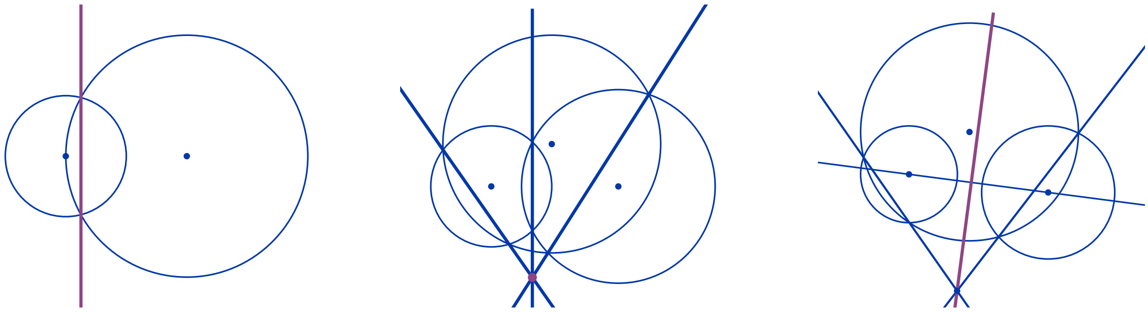


Power-of-a-point. Given a point P, a circle $\mathcal{C}$ with centre S and radius r, and a line through P intersecting the circle at A and B, we have

$$h(P,\mathcal{C}) = \|PS\| - r^2 = \|PA\|\|PB\|.$$

Radical axis. Given two circles, the straight line consisting of points that have equal power to both circles is called the **radical axis** of the two circles.

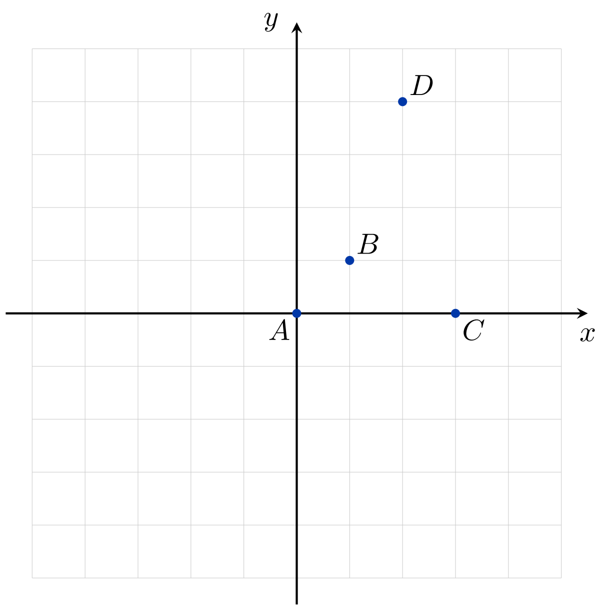
Radical center. The radical center of three circles is the unique point with equal power to all three circles.



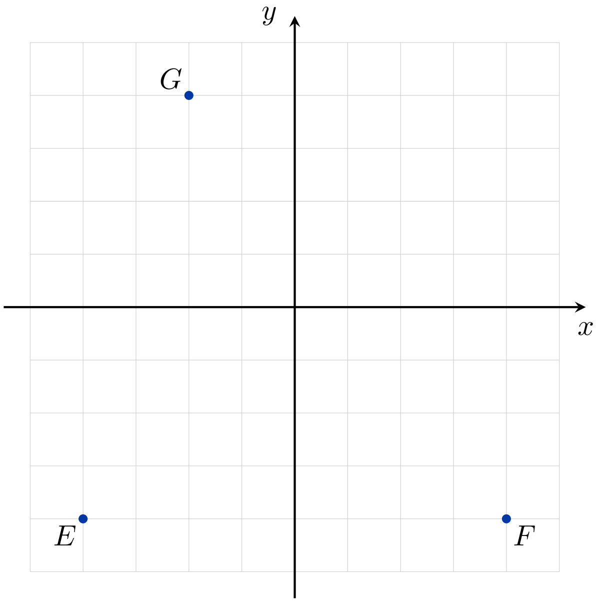
If the two circles intersect their radical axis passes through the intersections. If two circles do not intersect the radical line can be constructed by first constructing a third circle that intersects both, finding the radical center by constructing two of the radical lines and then drawing a line through the radical center perpendicular to the line between the centers of the first two circles.

Problem 24.

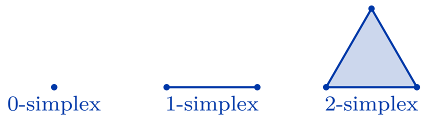
Hermes messenger service, Ltd. has distribution centres placed at $A(0,0)$, $B(1,1)$, $C(3,0)$ and $D(2,4)$. Divide the $[-5,5] \times [-5,5]$ square into service areas that ensure the fastest packet delivery.



Their competition, *Mercury post*, has the distribution centres located at $E(-4,-4)$, $F(4,-4)$ and $G(-2,4)$, but the center at E can only deliver within a 7 unit radius and the center at G only within a 6 unit radius. The center at F has more employees and uses bike messengers so they can deliver within an 10 unit radius. How should they split the service area?



A **k-simplex** is defined by its $k+1$ vertices $\langle v_0, \dots, v_k \rangle$. A 0-simplex is geometrically a point, a 1-simplex a line segment, a 2-simplex a triangle, a 3-simplex a tetrahedron, and so on. A simplex whose vertices form a subset of vertices of σ is called a **face** of σ.



A **simplicial complex** K is a set of simplices such that:

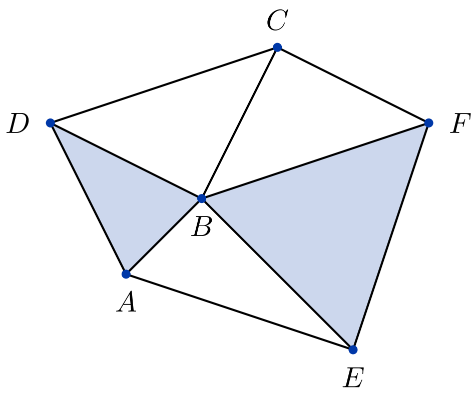
a. for each simplex $\sigma \in K$ all faces of σ are also in K and
b. for any two simplices $\sigma_1, \sigma_2 \in K$ their intersection is either empty or is a face of both σ_1 and σ_2.

Let $S \subset K$ be a subset of simplices in K.

- The **closure** of S, $\mathrm{cl}(S)$, is the smallest simplicial subcomplex of K that contains each simplex in S. It is obtained by repeatedly adding to S each face of every simplex in S.
- The **open star** of S, $\mathrm{st}(S)$, is the union of the stars $\mathrm{st}(\sigma)$ for all $\sigma \in S$. For a single simplex σ, the star of σ is the set of simplices having σ as a face. (The star of S is generally not a simplicial complex).
- The **link** of S, $\mathrm{lk}(S)$, is defined as

$$\mathrm{lk}(S) = \mathrm{cl}(\mathrm{st}(S)) \setminus \mathrm{st}(\mathrm{cl}(S)).$$

Problem 25.
Find the open stars $\mathrm{st}(A)$, $\mathrm{st}(AB)$ and the links $\mathrm{lk}(A)$, $\mathrm{lk}(AB)$ for the simplicial complex given below.



PROBLEM 26.
The simplicial complex K contains the following simplices:

$$\langle v_0\rangle, \langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle, \langle v_4\rangle, \langle v_0, v_1\rangle, \langle v_0, v_3\rangle, \langle v_1, v_3\rangle, \langle v_0, v_1, v_2\rangle.$$

a. Add any simplices that are missing from K.
b. Draw the Hasse diagram of K.
c. Find the open stars $\mathrm{st}(\langle v_1\rangle)$, $\mathrm{st}(\langle v_1, v_3\rangle)$ and the links $\mathrm{lk}(\langle v_2\rangle)$, $\mathrm{lk}(\langle v_0, v_3\rangle)$. Mark them on the Hasse diagram as well.

A **triangulated topological 2-dimensional manifold without boundary** is a 2-dimensional simplicial complex M, such that

- for each vertex v of M the link $\mathrm{lk}(v)$ is a triangulated S^1 (a connected graph with one cycle and no points of degree 1),
- for each edge uv of M the link $\mathrm{lk}(uv)$ is S^0 (a set of two disjoint vertices).

A 2-dimensional simplicial complex M **triangulated topological 2-dimensional manifold with boundary** if

- for each vertex v of M the link $\mathrm{lk}(v)$ is either a triangulated S^1 or a path (a connected tree with exactly two points of degree 1),
- for each edge uv of M the link $\mathrm{lk}(uv)$ is either S^0 or consists of a single point.

The vertices and edges of the first type are called **interior vertices** and **interior edges**, the vertices and edges of the second type are called **boundary vertices** and **boundary edges**.

The **Euler characteristic** χ of an n-dimensional simplicial complex K is

$$\chi(K) = c_0 - c_1 + c_2 - c_3 + \ldots + (-1)^n c_n,$$

where c_i is the number of simplices of dimension i.

For an orientable surface M of genus g with Euler characteristic $\chi(M)$ and b boundary components we have

$$\chi(M) = 2 - 2g - b.$$

For a non-orientable surface N of genus g with Euler characteristic $\chi(N)$ we have

$$\chi(N) = 2 - g.$$

PROBLEM 27.
For each of the following triangulations determine if it is a triangulation of a surface.

A: `[(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)]`

B: `[(1, 2, 3), (1, 2, 4), (2, 3, 5), (2, 3, 6), (3, 5, 7)]`
C: `[(1, 2, 3), (2, 3, 4), (3, 4, 5),`
`(4, 5, 6), (1, 5, 6), (1, 2, 6)]`
D: `[(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),`
`(1, 4, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),`
`(4, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),`
`(2, 3, 9), (3, 5, 9), (1, 3, 5)]`
E: `[(1, 2, 4), (2, 4, 6), (2, 3, 6), (3, 6, 8), (1, 3, 8),`
`(1, 5, 8), (4, 5, 6), (5, 6, 7), (6, 7, 8), (7, 8, 9),`
`(5, 8, 9), (4, 5, 9), (1, 5, 7), (1, 2, 7), (2, 7, 9),`
`(2, 3, 9), (3, 4, 9), (1, 3, 4)]`
F: `[(1, 2, 3), (1, 3, 4), (2, 3, 4), (4, 5, 6)]`
G: `[(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (2, 5, 6), (1, 2, 6)]`
H: `[(1, 3, 5), (1, 2, 6), (1, 5, 6), (1, 2, 4), (1, 3, 4),`
`(2, 3, 5), (2, 3, 6), (2, 4, 5), (3, 4, 6), (4, 5, 6)]`

a. Find the Euler characteristics for all of these simplicial complexes.
b. For each case check if the given triangulation belongs to a surface (a 2-dimensional triangulated manifold).
c. Find the number of boundary components for all of the surfaces.
d. For each of the surfaces determine if it is orientable or not.
e. Determine the genus of each orientable surface and the genus of non-orientable surfaces with no boundary.
f. Identify each of the surfaces.

CHAPTER 4

Vietoris-Rips and Čech complexes

The **Vietoris-Rips complex** $\mathrm{Rips}(S, r)$ is an abstract simplicial complex, such that
- the vertex set is S and
- a subset $\sigma \subseteq S$ is a simplex if and only if the diameter of σ is at most r.

A subset $\sigma \subseteq S$ is a simplex in $\mathrm{Rips}(S, r)$ if and only if

$$\overline{B}(x, \tfrac{r}{2}) \cap \overline{B}(y, \tfrac{r}{2}) \neq \emptyset$$

for all $x, y \in \sigma$.

The **Čech complex** $\mathrm{Cech}(S, r)$ is an abstract simplicial complex, such that
- the vertex set is S and
- a subset $\sigma \subseteq S$ is a simplex if and only if

$$\bigcap_{x \in \sigma} \overline{B}(x, r) \neq \emptyset.$$

If $r_1 \leq r_2$, then

$$\mathrm{Rips}(S, r_1) \subseteq \mathrm{Rips}(S, r_2) \quad \text{and} \quad \mathrm{Cech}(S, r_1) \subseteq \mathrm{Cech}(S, r_2).$$

For all r

$$\mathrm{Cech}(S, r) \subseteq \mathrm{Rips}(S, 2r) \subseteq \mathrm{Cech}(S, 2r)$$

and in Euclidean spaces

$$\mathrm{Rips}(S, r\sqrt{2}) \subseteq \mathrm{Cech}(S, r).$$

Problem 28. ⇩

Let

$$S = \{A(0,0), B(0.5,0.5), C(0,1), D(1,2), E(1.5,1.5), F(2,1.5), G(2.5,1), H(2,0)\} \subset \mathbb{R}^2.$$

Build the Vietoris-Rips complex $\mathrm{Rips}(S, r)$ for

a. $r = 1$,
b. $r = 1.2$,
c. $r = 1.75$.

In each case list all the simplices and determine its dimension. Assuming there is a sensor placed at each point of S and all sensors can detect points that are at distance 1.75 or less, is the area covered by the sensors connected? Does it contain any holes?

Problem 29. ⇩

Let

$$S = \{A(0,0), B(0.5,0.5), C(0,1), D(1,2), E(1.5,1.5), F(2,1.5), G(2.5,1), H(2,0)\} \subset \mathbb{R}^2.$$

Build the Čech complex $\mathrm{Cech}(S, r)$ for

a. $r = 0.5$,
b. $r = 0.6$,
c. $r = 0.875$.

In each case list all the simplices and determine its dimension.

The **Levenshtein distance** between two words (strings or sequences of letters) is defined as the minimum number of edits needed to transform one string into the other. The operations allowed are

- insertion of a single character at any position,
- deletion of a single character at any position or
- substitution of a single character at any position for any other character.

Problem 30.
Using the Levenshtein distance on the set of words

$$S = \{\texttt{SONCE}, \texttt{SENCE}, \texttt{SREDA}, \texttt{VRAG}, \texttt{SRENJ}\},$$

build the filtration of the Vietoris-Rips complexes

$$\mathrm{Rips}(S, 1) \subseteq \mathrm{Rips}(S, 2) \subseteq \mathrm{Rips}(S, 3) \subseteq \mathrm{Rips}(S, 4) \subseteq \mathrm{Rips}(S, 5) \subseteq \mathrm{Rips}(S, 6).$$

To speed things up you can use an online Levenshtein distance calculator or write your own version.

Problem 31.
Let $S = \{A(0, 0), B(1, 1), C(-2, 3), D(2, 3), E(4, 2), F(3, -1)\}$.

a. List all the maximal simplices of the Vietoris-Rips complex with $R = 2.4$.
b. List all the maximal simplices of the Vietoris-Rips complex with $R = 4.1$.
c. List all the maximal simplices of the Čech complex with $r = 1.2$.
d. List all the maximal simplices of the Čech complex with $r = 2.05$.

CHAPTER 5

Simplicial homology

For a simplicial complex K and simplices $\tau < \sigma$ we say that τ is a **free face** of σ if it is not contained in any other simplex of K.

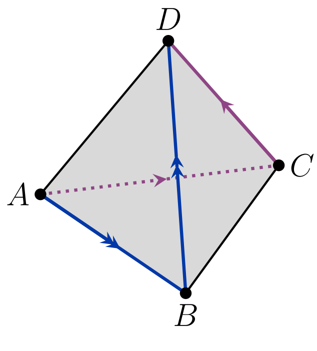
If τ is a free face of σ in a simplicial complex K, then we can collapse the pair (σ, τ) by removing them from the list of simplices. The resulting simplicial complex $K \setminus \{\sigma, \tau\}$ is homotopy equivalent to K.

PROBLEM 32.
The simplicial complexes X and Y are given as lists of simplices:

$$\begin{aligned} X &= \{A, B, C, AB, AC, BC\}, \\ Y &= \{A, B, C, D, AB, AD, BC, CD\}. \end{aligned}$$

a. Construct the cones CX and CY by listing all the simplices.
b. Find the sequences of collapses that simplify CX and CY as much as possible.
c. Is CX a collapsible complex for all X?

PROBLEM 33.
Let $X = \Delta^3$ be the standard 3-simplex (tetrahedron) with vertices A, B, C and D. We obtain Y by identifying the edges AB and BD and the edges AC and CD (preserving the ordering of vertices). Show that Y collapses onto a Klein bottle.



Given a map $f\colon U \to V$ between vector spaces U and V we define the **kernel** of f as

$$\ker f = \{x \in U \mid f(x) = 0\} \subset U$$

and the **image** of f as

$$\mathrm{im} f = \{f(x) \in V \mid x \in U\} \subset V.$$

If the map f is represented by a matrix F, then the kernel of f is the nullspace of F and the image of f is the column space of F.

Given a sequence

$$\dots \to \mathcal{C}_{n+1} \xrightarrow{\partial_{n+1}} \mathcal{C}_n \xrightarrow{\partial_n} \mathcal{C}_{n-1} \to \dots$$

of vector spaces over a field F such that $B_n = \mathrm{im}\partial_{n+1} \subseteq \ker \partial_n = Z_n$, we can compute the quotients

$$H_n = \frac{\ker \partial_n}{\mathrm{im}\partial_{n+1}} = \frac{Z_n}{B_n}.$$

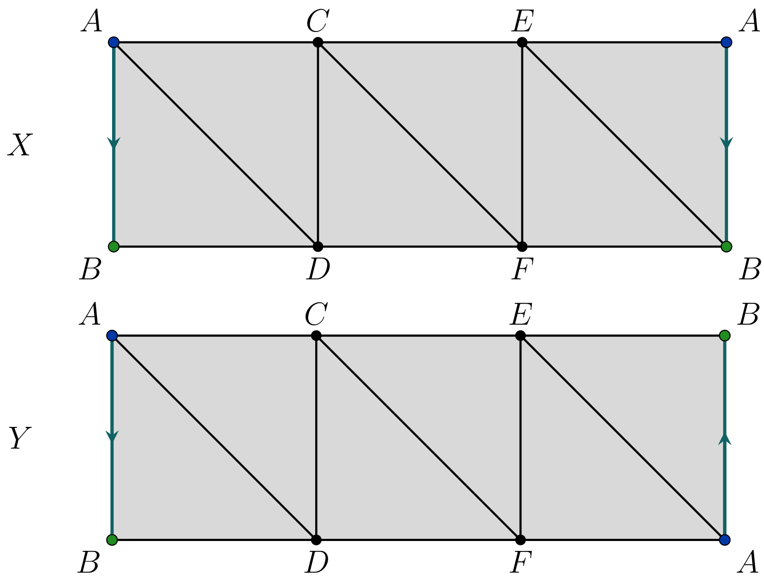
If we obtain the chain groups $\mathcal{C}_n$ from a simplicial structure on X, we say $H_n = H_n(X;F)$ is the n^{th} **simplicial homology group** of X with coefficients in F.
The numbers $b_n = \mathrm{rank}(H_n)$ are called the **Betti numbers** of X. We have

$$\chi(X) = b_0 - b_1 + b_2 - b_3 + \cdots.$$

If $H_n = \mathbb{Z}^k \oplus \mathbb{Z}_{p_1}^{k_1} \oplus \cdots \oplus \mathbb{Z}_{p_m}^{k_m}$, then $b_n = \mathrm{rank}(H_n) = k$.

Problem 34.

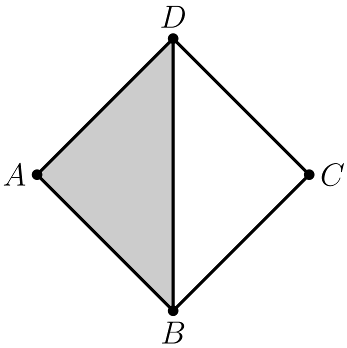
Given the following triangulations of the cylinder X and the Moebius band Y, find a sequence of elementary collapses that simplifies them as much as possible, then compute the homology groups $H_*(X)$ and $H_*(Y)$.



Problem 35.

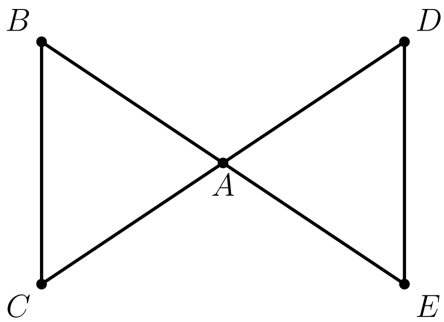
For the simplicial complex X in the figure below

a. write down the chain groups $\mathcal{C}_n$,
b. determine the boundary homomorphisms $\partial_n \colon \mathcal{C}_n \to \mathcal{C}_{n-1}$,
c. find the cycles $Z_n = \ker\partial_n$,
d. find the boundaries $B_n = \mathrm{im}\partial_n$,
e. determine the simplicial homology groups with $\mathbb{Z}$ coefficients, $H_n(X;\mathbb{Z})$,
f. determine the simplicial homology groups with $\mathbb{Z}_2$ coefficients, $H_n(X;\mathbb{Z}_2)$,
g. determine the Betti numbers of X and
h. compute the Euler characteristic of X.



Problem 36.
For the simplicial complex X in the figure below

a. write down the chain groups $\mathcal{C}_n$,
b. determine the boundary homomorphisms $\partial_n \colon \mathcal{C}_n \to \mathcal{C}_{n-1}$,
c. find the cycles $Z_n = \ker\partial_n$,
d. find the boundaries $B_n = \mathrm{im}\partial_n$,
e. determine the simplicial homology groups with $\mathbb{Z}$ coefficients, $H_n(X;\mathbb{Z})$,
f. determine the simplicial homology groups with $\mathbb{Z}_2$ coefficients, $H_n(X;\mathbb{Z}_2)$,
g. determine the Betti numbers of X and
h. compute the Euler characteristic of X.



Problem 37.
Calculate the following products:

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$,

c. $\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Identify which of these swap two rows/columns, add a multiple of a row/column to another row/column and multiply a row/column with a number.

The **reduced row echelon form** of a matrix D is a matrix D_R such that:

- all non-zero rows of D_R are above all zero rows,
- the leftmost non-zero entry in every row is 1,
- the leftmost 1 of every row is strictly to the left of the leftmost 1 in the next row,
- the leftmost 1 of every row is the only non-zero entry in its column.

The reduced row echelon form is unique. To obtain it, we are allowed to

- swap any two rows,
- multiply a row by a non-zero constant,
- add a multiple of one row to another.

The **reduced column echelon form** of a matrix D is a matrix D_C such that:

- all non-zero columns of D_C are to the left all zero columns,
- the topmost non-zero entry in every column is 1,
- the topmost 1 of every column is strictly above of the topmost 1 in the next column,
- the topmost 1 of every column is the only non-zero entry in its row.

The reduced column echelon form is unique. To obtain it, we are allowed to

- swap any two columns,
- multiply a column by a non-zero constant,
- add a multiple of one column to another.

Define the matrices

- $P_{i,j}$ (obtained from the identity matrix by swapping the i^{th} and the j^{th} row),
- M_i (obtained from the identity matrix by setting the element at the crossing of the i^{th} row and i^{th} column to k instead of 1), and
- $A_{i,j}$ (obtained from the identity matrix by changing the element in the i^{th} row and j^{th} column from 0 to k, assuming $i \neq j$).

Then multiplying from the left by

- $P_{i,j}$ swaps the i^{th} and the j^{th} row,
- M_i multiplies the i^{th} row by k,
- $A_{i,j}$ has the effect of adding the j^{th} row to the i^{th} row.

Multiplying from the right by

- $P_{i,j}$ swaps the i^{th} and the j^{th} column,
- M_i multiplies the i^{th} column by k,
- $A_{i,j}$ has the effect of adding the i^{th} column to the j^{th} column.

Given a sequence

$$\mathcal{C}_{n+1} \xrightarrow{D_{n+1}} \mathcal{C}_n \xrightarrow{D_n} \mathcal{C}_{n-1},$$

we wish to compute

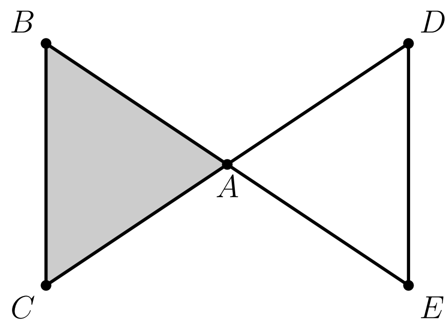
$$H_n = \frac{\ker D_n}{\mathrm{im} D_{n+1}}.$$

To do so, we need to have the same basis for $\ker D_n$ and for imD_{n+1}.
We achieve this by first performing a column-reduction by right matrix multiplication on D_n to get a transformation matrix P. Then we calculate the products D_nP and $P^{-1}D_{n+1}$ and perform a row reduction on the non-zero rows of $P^{-1}D_{n+1}$ to obtain a matrix Q.
The zero-columns of D_nPQ^{-1} are the basis for $\ker D_n$, the non-zero rows of $QP^{-1}D_{n+1}$ are the basis for $\mathrm{im}D_{n+1}$.

Problem 38.

For the simplicial complex X in the figure below

a. write down the chain groups $\mathcal{C}_2$, $\mathcal{C}_1$ and $\mathcal{C}_0$,
b. write down the matrices D_n for the boundary homomorphisms $\partial_n\colon \mathcal{C}_n \to \mathcal{C}_{n-1}$ for $n = 0, 1, 2, 3$,
c. compute the homology groups,
d. collapse the free faces and determine how this changes the boundary matrices.

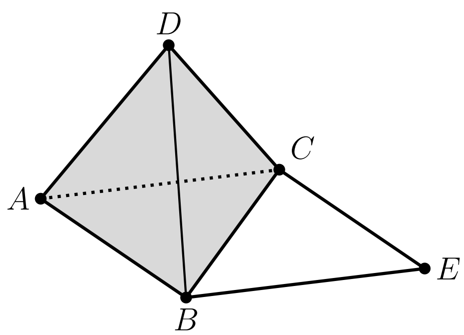


Problem 39.

Write down the corresponding chain groups $\mathcal{C}_2$, $\mathcal{C}_1$ and $\mathcal{C}_0$ for the Moebius strip (use the triangulation from Problem 34) and the matrices D_n for the boundary homomorphisms $\partial_n\colon \mathcal{C}_n \to \mathcal{C}_{n-1}$. Compute the homology.

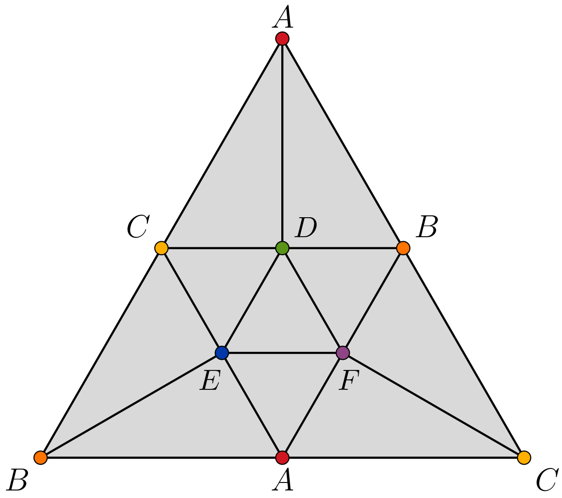
Problem 40.

Let X be a simplicial complex with maximal simplices ABC, ABD, ACD, BCD, BE and CE. Compute $H_*(X)$.



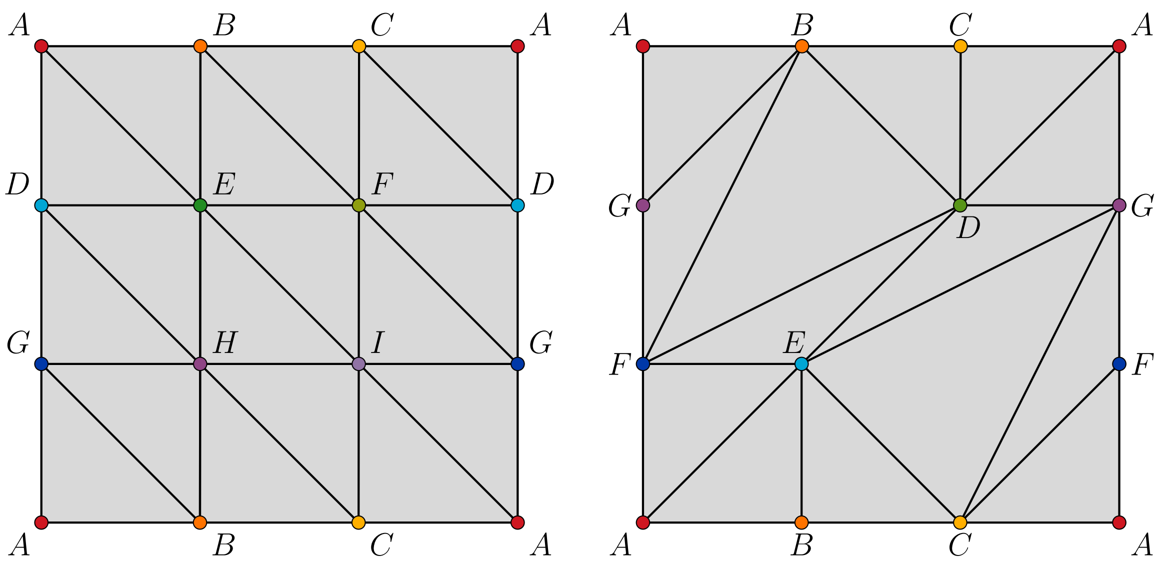
Problem 41.

Write down the chain complexes $\mathcal{C}_2$, $\mathcal{C}_1$ and $\mathcal{C}_0$ for the given triangulation of the real projective plane $P = \mathbb{R}P_2$. Find the matrices D_2 and D_1 for the boundary homomorphisms $\partial_2\colon \mathcal{C}_2 \to \mathcal{C}_1$ and $\partial_1\colon \mathcal{C}_1 \to \mathcal{C}_0$. Compute the homology groups.



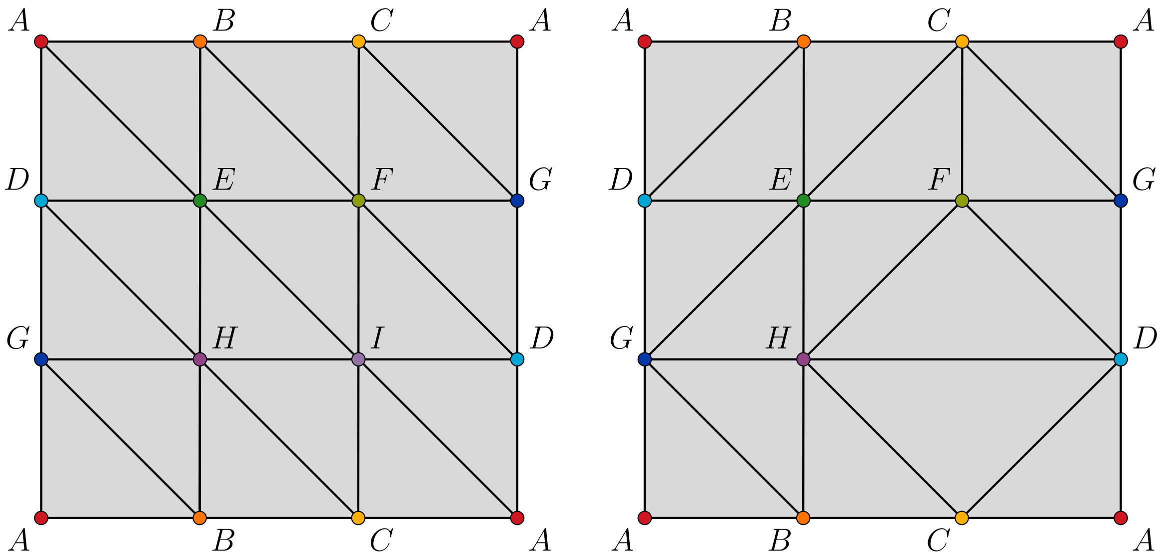
Problem 42.

The minimal triangulation of the torus T has 7 vertices, 21 edges and 14 faces. Write down the corresponding chain complexes $\mathcal{C}_2$, $\mathcal{C}_1$ and $\mathcal{C}_0$ and the matrices D_2 and D_1 for the boundary homomorphisms $\partial_2 \colon \mathcal{C}_2 \to \mathcal{C}_1$ and $\partial_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$. Use your favourite computational topology software to compute the homology groups.



Problem 43.

The minimal triangulation of the Klein bottle has 8 vertices, 24 edges and 16 faces, but finding one that consists of 9 vertices, 27 edges and 18 faces is much easier. Choose one and write down the corresponding chain complexes $\mathcal{C}_2$, $\mathcal{C}_1$ and $\mathcal{C}_0$. Find the matrices D_2 and D_1 for the boundary homomorphisms $\partial_2 \colon \mathcal{C}_2 \to \mathcal{C}_1$ and $\partial_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$ and compute the homology.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CHAPTER 6

Persistence

For a simplicial complex K a **filtration** of K is a sequence of simplicial complexes

$$K_0 \leq K_1 \leq \ldots \leq K_n = K.$$

We can obtain filtrations from the skeleta of K (filtered by dimension), the Vietoris-Rips or Cech complexes (filtered by the radius), from discrete Morse functions, etc.
A **critical event** in a filtration is when the homotopy type of K_{i+1} is different from the homotopy type of K_i.

Given a filtration

$$K_0 \leq K_1 \leq \ldots \leq K_n = K,$$

the **persistent homology group** H_p^{jt} of the pair (j, t) is the p^{th} homology group of the complex K_j, computed in K_t:

$$H_p^{jt} = \frac{Z_p(K_j)}{B_p(K_t) \cap Z_p(K_j)}.$$

The corresponding **persistent Betti number** is $b_p^{jt} = \text{rank} H_p^{jt}$ and is equal to the number of independent non-trivial homology classes of $H_p(K_j)$ that are still non-trivial in $H_p(K_t)$.
A homology class $\gamma \in H_p(K_i)$ **is born** in K_i, if $\gamma \notin H_p^{i-1,j}$. A homology class that was born in K_i, **dies** in K_j for some $j > i$, if it is non-trivial in $K_{i+1}, \ldots, K_{j-1}$ and becomes a boundary in K_j. The **persistence** of the class γ is the interval between the birth and death of γ: $\text{pers}(\gamma) = [i, j)$. If γ is born and never dies then its persistence is $[i, \infty)$ and γ is a generator of the homology of K.

A **barcode** is a diagram that contains for each class γ with persistence $[i, j)$ an interval (a bar) from i to j.
A **persistence diagram** is obtained by associating to each class γ with persistence $[i, j)$ a point in the plane with coordinates (i, j). If the multiplicity

$$\mu_p^{ij} = (b_p^{i,j-1} - b_p^{ij}) - (b_p^{i-1,j-1} - b_p^{i-1,j})$$

is greater than 1, we add it as a label to the point (i, j). If the persistence of γ is $[i, \infty)$, we draw a vertical ray starting at (i, i). Finally, we also include the diagonal

$$\Delta = \{(x, x) \mid x \in \mathbb{R}\}.$$

PROBLEM 44. ⇩

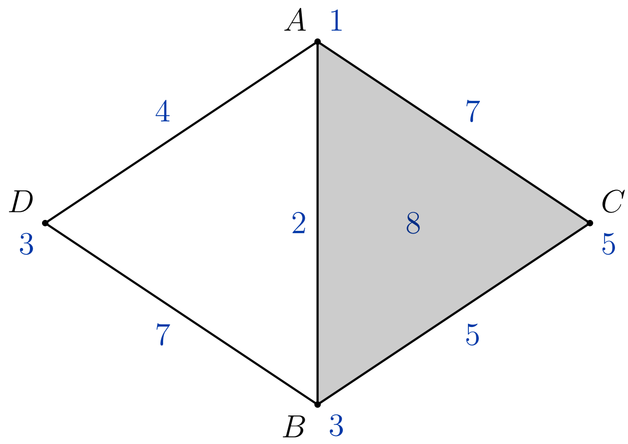
Let K be the simplicial complex

$$K = \{A,\, B,\, C,\, D,\, AB,\, AC,\, AD,\, BC,\, BD,\, ABC\}.$$

The function f assigns the values 1, 3, 5, 3, 2, 7, 4, 5, 7, 8 to the simplices of K (in this order).

a. Write down the filtration of the complex K and draw the sublevel complexes.

b. Determine the critical events in dimensions 0 and 1.
c. Plot the barcodes in dimensions 0 and 1.
d. Plot the persistance diagrams in dimensions 0 and 1.



Problem 45.

For the set of points $S = \{(0,0), (2,0), (1,2), (2,-2)\}$ construct the (closed) Čech complexes $\mathrm{Cech}(S, r)$ with $r \in \{0, 1, 1.2, 1.5, 2, 2.5\}$. These complexes determine a filtration

$$\mathrm{Cech}(S,0) \leq \mathrm{Cech}(S,1) \leq \mathrm{Cech}(S,1.2) \leq \mathrm{Cech}(S,1.5) \leq \mathrm{Cech}(S,2) \leq \mathrm{Cech}(S,2.5).$$

a. Draw each stage of the filtration for $r \in \{1, 1.2, 1.5, 2, 2.5\}$ and determine the critical events.
b. Plot the barcodes in dimensions 0, 1 and 2.
c. Plot the persistance diagrams in dimensions 0, 1 and 2.

Problem 46.

Let K be the simplicial complex

$$\begin{aligned} K &= \{A, B, C, D, E, \\ &\quad AB, AC, AD, AE, BC, BD, BE, CD, \\ &\quad ABC, ABD, ACD, BCD\}. \end{aligned}$$

a. Write down the filtration of the complex K by sublevel complexes, adding one simplex at a time, ordered by dimension and then lexicographically.
b. Plot the barcodes in dimensions 0, 1 and 2.
c. Plot the persistance diagrams in dimensions 0, 1 and 2.

Problem 47.

Let K be the simplicial complex

$$\begin{aligned} K &= \{A, B, C, D, E, F, \\ &\quad AE, AF, BD, BF, CD, CE, DE, DF, EF, \\ &\quad AEF, BDF, DEF\}. \end{aligned}$$

A filtration on K is given by assigning the following values to the simplices of K:

A	B	C	D	E	F
1	2	2	4	4	5

AE	AF	BD	BF	CD	CE	DE	DF	EF
5	3	3	6	5	3	7	7	8

AEF	BDF	DEF
7	7	9

a. Write down the filtration of the complex K by sublevel complexes and draw them.
b. Determine the critical events in dimensions 0 and 1.
c. Plot the barcodes in dimensions 0 and 1.
d. Plot the persistance diagrams in dimensions 0 and 1.

In L_∞-norm the distance between points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is defined as
$$\|x - y\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Problem 48.
For a point $A = (x, y)$ determine the point $D = (d, d)$ on the diagonal
$$\Delta = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$$
such that the distance $\|A - D\|_\infty$ is minimal. Express this distance in terms of x and y.

The **bottleneck distance** between persistence diagrams X and Y is defined as
$$W_\infty(X, Y) = \inf_{\eta\colon X \to Y} \left(\sup_{x \in X} \|x - \eta(x)\|_\infty \right).$$

Problem 49.
A persistence diagram consists of the diagonal Δ and finitely many points in the extended plane $\mathbb{R} \times \overline{\mathbb{R}}$ above the diagonal.

Let $A = (1, 3)$, $B = (2, 4)$, $C = (1, 2)$ $D = (2, 5)$ and $E = (1, \infty)$. For each of the pairs X_i, Y_i of persistent diagrams given below

- find all bijections $\eta\colon X_i \to Y_i$,
- determine $\|x - \eta(x)\|_\infty$ for each bijection and for all $x \in X_i$ and
- calculate the bottleneck distance $W_\infty(X_i, Y_i)$.

a. $X_1 = \Delta \cup \{A\}$, $Y_1 = \Delta \cup \{B\}$,
b. $X_2 = \Delta \cup \{A, B\}$, $Y_2 = \Delta \cup \{C\}$,
c. $X_3 = \Delta \cup \{A, B\}$, $Y_3 = \Delta \cup \{C, D\}$,
d. $X_4 = \Delta \cup \{A, E\}$, $Y_4 = \Delta \cup \{C\}$.

The bottleneck distance only takes into account the furthest pair of corresponding points for each bijection. To fix this shortcoming we define the **Wasserstein distance** for all $q \in \mathbb{R}$:
$$W_q(X, Y) = \left(\inf_{\eta\colon X \to Y} \sum_{x \in X} \|x - \eta(x)\|_\infty^q \right)^{\frac{1}{q}}.$$

Problem 50.
For X_i and Y_i from the previous problem calculate the Wasserstein distances

a. $W_1(X_1, Y_1)$,
b. $W_2(X_1, Y_1)$,
c. $W_1(X_2, Y_2)$,
d. $W_2(X_2, Y_2)$,
e. $W_1(X_3, Y_3)$,
f. $W_2(X_3, Y_3)$.

Given a simplicial complex $K = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, we can define a matrix D by

$$D_{i,j} = \begin{cases} 1; & \sigma_i < \sigma_j, \\ 0; & \text{otherwise.} \end{cases}$$

Define

$$\text{low(j)} = \text{row index of the lowest 1 in column } j.$$

We use column operations to reduce D to R. The matrix R is reduced if $\text{low}(j) \neq \text{low}(j_0)$ for all $j \neq j_0$. The algorithm for obtaining R from D is:

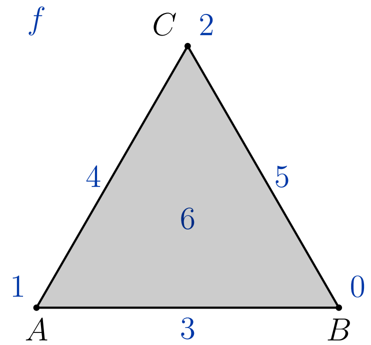
```
R = D
for j = 1 to m:
    while there exists j0 < j with low(j0) = low(j):
        add column R[:, j0] to column R[:, j]
```

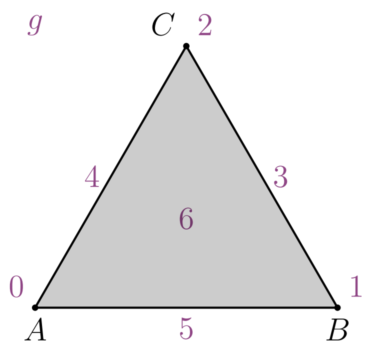
Problem 51.

Two different monotonic functions are given on the simplicial complex X:

$$\begin{aligned} f &= \{(A,1), (B,0), (C,2), (AB,3), (AC,4), (BC,5), (ABC,6)\}, \\ g &= \{(A,0), (B,1), (C,2), (AB,5), (AC,4), (BC,3), (ABC,6)\}. \end{aligned}$$

a. Order the simplices of X into a sequence $\sigma_1, \ldots, \sigma_7$ so that $f(\sigma_i) < f(\sigma_j)$ if $i < j$. Do the same for g.
b. Use this sequences to create filtrations of subcomplexes and draw the barcode diagrams in dimensions $p = 0$ and $p = 1$.
c. Draw the corresponding persistence diagrams $\text{Dgm}_p(f)$ and $\text{Dgm}_p(g)$ in dimensions $p = 0$ and $p = 1$.
d. Calculate the bottleneck distances between the persistence diagrams $\text{Dgm}_p(f)$ and $\text{Dgm}_p(g)$ for $p = 0$ and $p = 1$.
e. Construct the boundary matrices D_f and D_g from the two filtrations.
f. Use the matrix reduction to calculate persistence.





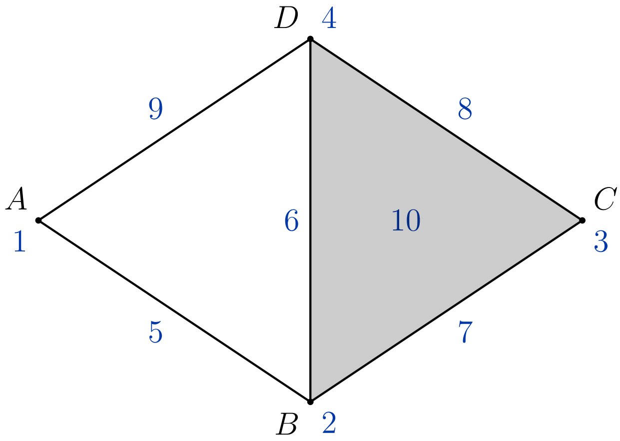
Problem 52.

A monotonic function f is given on the simplicial complex X:

A	B	C	D	AB	AD	BC	BD	CD	BCD
1	2	3	4	5	9	7	6	8	10

a. Order the simplices of X into a sequence $\sigma_1, \ldots, \sigma_{10}$ so that $f(\sigma_i) < f(\sigma_j)$ if $i < j$.
b. Use this sequence to create a filtration of subcomplexes and draw the barcode diagrams in dimensions $p = 0$ and $p = 1$.
c. Draw the corresponding persistence diagrams $\text{Dgm}_p(f)$ in dimensions $p = 0$ and $p = 1$.

d. Construct the boundary matrix D_f.
e. Use the matrix reduction to calculate persistence.



Problem 53.

Use the algorithm given above to find the matrix R for Problem 46. Can you find the persistence classes in the reduced matrix?

CHAPTER 7

Morse theory

A function $F\colon K \to \mathbb{R}$ is a **discrete Morse function** on a simplicial complex K if for every $\alpha^{(p)} \in K$

- the set $\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\}$ contains at most one element and
- the set $\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\}$ contains at most one element.

It can be shown that at least one of these sets must be empty. A simplex $\alpha^{(p)}$ is called **critical** if both sets are empty.

A **discrete vector field** V on a simplicial complex K is a collection of pairs

$$\{\alpha^{(p)} < \beta^{(p+1)}\}$$

of simplices in K such that each simplex is in at most one pair of V. We can represent V graphically by drawing an arrow from the centre of α to the centre of β for each pair $\{\alpha < \beta\}$ in V. Every Morse function F defines a discrete vector field V_F with $(\alpha^{(p)}, \beta^{(p+1)}) \in V_F$ if and only if $F(\alpha) \geq F(\beta)$.

A V**-path** is a sequence of simplices

$$\alpha_0^{(p)},\, \beta_0^{(p+1)},\, \alpha_1^{(p)},\, \beta_1^{(p+1)},\, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)},\, \alpha_{r+1}^{(p)}$$

such that for each $i \in \{0, 1, \ldots, r\}$ the pair (α_i, β_i) is in V and $\beta_i > \alpha_{i+1}$ with $\alpha_{i+1} \neq \alpha_i$. The path is **closed** if $\alpha_0 = \alpha_{r+1}$.
A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed V-paths.
If V is the gradient vector field of a discrete Morse function F, then

$$F(\alpha_0) \geq F(\beta_0) \geq F(\alpha_1) \geq F(\beta_1) \geq \ldots \geq F(\beta_r) \geq F(\alpha_{r+1}).$$

If F is a discrete Morse function on K with critical simplices $\beta^{(p+1)}$ and $\alpha^{(p)}$ and there is exactly one gradient path from $\partial\beta$ to α, then there is another Morse function G on K with the same critical simplices except α and β. The gradient vector field of G is equal to that of F except along the unique gradient path from β to α, where the field is reversed.

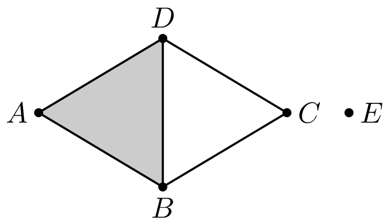
Problem 54.

For a given simplicial complex K the functions F and G are defined by the values in the following array.

σ	A	B	C	D	E	AB	AD	BC	BD	CD	ABD
$F(\sigma)$	1	3	5	5	2	2	7	4	4	6	6
$G(\sigma)$	3	2	0	3	2	6	4	1	4	1	5

a. Show that F and G are discrete Morse functions on K.
b. Determine the critical simplices and draw the corresponding vector fields.

c. Find all non-trivial gradient paths and use cancellation to obtain new vector fields with the minimal possible number of critical simplices.
d. Construct a discrete Morse function H on K that has $c_2 = 1$, $c_1 = 2$ and $c_0 = 2$. Then cancel all possible pairs of critical simplices in the corresponding vector field to minimize their number.



To construct a discrete Morse function F on a 1-dimensional simplicial complex K:

- find a spanning tree of K,
- pick a vertex v_0 and set $F(v_0) = 0$,
- for all other vertices v in K let $F(v)$ be twice the distance to the root v_0 along the unique path in the spanning tree,
- for every edge uv contained in the spanning tree let $F(uv) = \frac{1}{2}(F(u) + F(v))$,
- for all edges uv not in the spanning tree choose $F(uv)$ greater than the maximal value of F on the spanning tree.

All edges not in the spanning tree and v_0 are critical simplices of F.

To construct a discrete Morse function F on a simplicial complex K from a given discrete vector field V:

- let $F(\sigma) = \dim(\sigma)$ for all critical simplices σ of V,
- for every directed path starting in a critical p-simplex assign descending values from the interval $(p, p-1)$ until reaching a simplex τ with $F(\tau)$ already assigned, then if $F(\tau)$ is bigger than the value of F on the previous simplex in the path, assign a new (lower) value from $(p, p-1)$ to τ and appropriate new values along all directed paths continuing from τ.

If L is a sub-complex of K, then any discrete Morse function F on L can be extended to a discrete Morse function G on K by

$$G(\sigma) = \begin{cases} F(\sigma), & \sigma \in L, \\ \dim(\sigma) + \max_{\tau \in L} F(\tau), & \sigma \in K \setminus L \end{cases}$$

This is not very efficient since every simplex in $K \setminus L$ is critical. If K is collapsible to L, we can improve on this. If $K = L \cup \{\alpha^{(p)}, \beta^{(p+1)}\}$ and α is a free face of β, then we can define

$$G(\sigma) = \begin{cases} F(\sigma), & \sigma \in L, \\ \max_{\tau \in L} F(\tau) + 1, & \sigma = \beta, \\ \max_{\tau \in L} F(\tau) + 2, & \sigma = \alpha. \end{cases}$$

It is easy to see that G is a Morse function on K that extends F and the two simplices α and β are not critical. Using this we can extend any Morse function one collapsible pair at a time, only adding critical simplices when we run out of collapsible pairs (ie. when homology of the complex changes).

Let K be a simplicial complex of dimension n and choose a discrete gradient vector field on K. For each p let M_p be the vector space with $\mathbb{Z}_2$ coefficients, spanned by critical p-simplices. The **Morse chain complex** is the sequence

$$0 \to M_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{p+1}} M_p \xrightarrow{\partial_p} M_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} M_0 \to 0.$$

The boundary homomorphism $\partial_p \colon M_p \to M_{p-1}$ on critical p-simplex σ is given by

$$\partial_p(\sigma^p) = \sum_i \alpha_i \tau_i^{p-1},$$

where α_i is the number of gradient paths (mod 2) starting from the boundary of σ and ending in τ_i. This is extended to chains as

$$\partial_p\left(\sum_i a_i \sigma_i\right) = \sum a_i \partial_p(\sigma_i).$$

If $Z_p(M) = \ker \partial_p$ and $B_p(M) = \mathrm{im}\partial_{p+1}$, then the **Morse homology** of the simplicial complex K with $\mathbb{Z}_2$ coefficients is

$$H_p(M;\mathbb{Z}_2) = \frac{Z_p(M)}{B_p(M)}.$$

The Morse homology groups are isomorphic to the simplicial homology groups, $H_p(M;\mathbb{Z}_2) \cong H_p(K;\mathbb{Z}_2)$, and the Betti numbers of K are $b_p(K;\mathbb{Z}_2) = \mathrm{rank} H_p(M;\mathbb{Z}_2)$.

Let c_p denote the number of critical simplices of K in dimension p and b_p the Betti numbers of K and let $n = \dim K$.

The Weak Morse inequalities.

For $p = 0, 1, 2, \ldots, n$ we have $c_p \geq b_p$ and

$$\chi(K) = c_0 - c_1 + c_2 - \ldots + (-1)^n c_n = b_0 - b_1 + b_2 - \ldots + (-1)^n b_n.$$

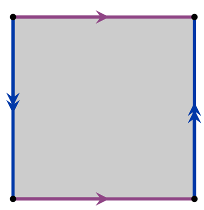
The Strong Morse inequalities.

For $p = 0, 1, 2, \ldots, n, n+1$ we have

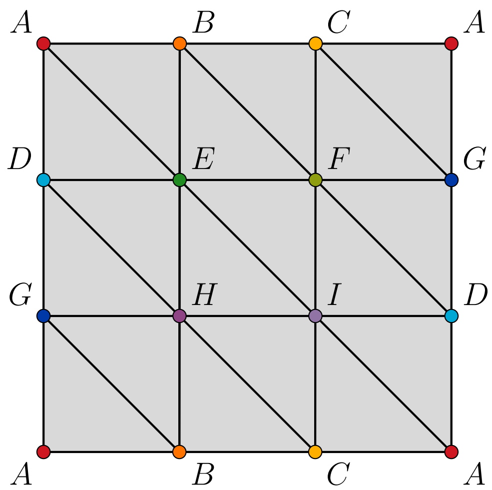
$$c_p - c_{p-1} + c_{p-2} - \ldots + (-1)^p c_0 \geq b_p - b_{p-1} + \ldots + (-1)^p b_0.$$

Problem 55.

Recall that the Klein bottle K is obtained by gluing two opposite sides of the square to obtain the cylinder and then gluing the other two sides by adding a twist (so that the arrows match).



One possible triangulation of the Klein bottle is given below.

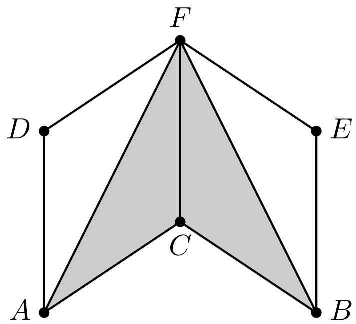


a. Write down a 1-cycle in K that is a boundary. Write down a 1-cycle in K that is not a boundary. Is there a 2-cycle in K? Is it a boundary?
b. Construct a Morse function F on K, draw the corresponding vector field and then cancel all possible pairs of critical simplices to minimize the number of critical simplices, or try drawing an optimal gradient vector field without constructing the function first.
c. Determine the number c_i of critical simplices of dimension i and compute the Euler characteristic $\chi(K)$.
d. What are the Betti numbers of K with $\mathbb{Z}_2$ coefficients? Is there a connection between the Betti numbers and the numbers c_i?

Problem 56.

For the 1-dimensional simplicial complex X given below, values are assigned to the vertices as follows:

A	B	C	D	E	F
0	0	2	4	4	6



a. Come up with an algorithm for extending the values at vertices to a discrete Morse function F on X.
b. Construct the corresponding gradient vector field V_F on X. How many critical simplices does it have? Compute the Euler characteristic of X.
c. Cancel as many pairs of critical simplices as possible.

Problem 57.

a. Let X be a simplicial complex. Show that the minimum of a discrete Morse function on X must occur at a vertex.

b. Suppose M is a triangulated surface without boundary. Show that the maximum of a discrete Morse function on M must occur on a 2-simplex.
c. Find an example of a simplicial complex X and a discrete Morse function F on X, such that the maximum of F does not occur on the top-dimensional simplex of X.

To compute Morse homology with $\mathbb{Z}$ coefficients let M_p be the vector space with $\mathbb{Z}$ coefficients, spanned by critical p-simplices with the corresponding Morse chain complex

$$0 \to M_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{p+1}} M_p \xrightarrow{\partial_p} M_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} M_0 \to 0.$$

We define an inner product $\langle \cdot\, , \cdot \rangle$ on M_p on generators of M_p as

$$\langle \sigma_1, \sigma_2 \rangle = \begin{cases} 0, & \sigma_1 \neq \sigma_2, \\ 1, & \sigma_1 = \sigma_2, \end{cases}$$

and extend it linearly to all elements of M_p.

For a gradient path

$$\gamma \colon \sigma_0,\ \tau_0,\ \sigma_1,\ \tau_1,\ \ldots,\ \tau_{r-1},\ \sigma_r,$$

we define the **multiplicity** of γ, $m(\gamma)$, as

$$m(\gamma) = \prod_{i=0}^{r-1} \langle \partial\tau_i, \sigma_i \rangle \langle \partial\tau_i, \sigma_{i+1} \rangle.$$

The multiplicity of γ is 1 if the orientation of σ_r agrees with the orientation obtained by sliding σ_0 along γ and -1 if the orientation is reversed. For any 1-path γ we have $m(\gamma) = 1$. The multiplicity of the constant path is also 1.
Let $\Gamma(\sigma, \sigma')$ denote the set of all gradient paths from σ to σ'. For critical simplices $\tau^{(p+1)}$ and $\sigma^{(p)}$ we define

$$\langle \partial\tau, \sigma \rangle = \sum_{\sigma' < \tau} \left(\langle \partial\tau, \sigma' \rangle \sum_{\gamma \in \Gamma(\sigma', \sigma)} m(\gamma) \right).$$

If $M_{p+1} = \langle \tau_1, \ldots, \tau_n \rangle$ and $M_p = \langle \sigma_1, \ldots, \sigma_m \rangle$, then the entries of the boundary matrix $D_p \colon M_{p+1} \to M_p$ are

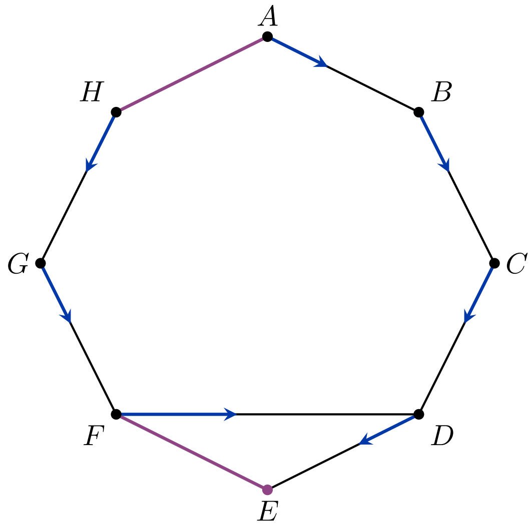
$$(D_p)_{i,j} = \langle \partial\tau_i, \sigma_j \rangle.$$

We obtain Morse homology groups with $\mathbb{Z}$ coefficients as

$$H_p(M; \mathbb{Z}) = \frac{\ker D_p}{\mathrm{im} D_{p+1}}.$$

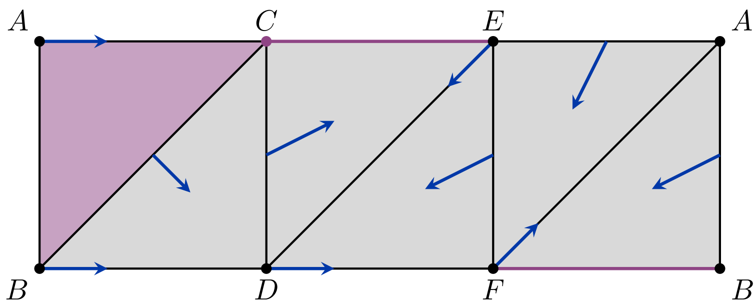
Problem 58.

Use the given discrete vector field to compute the homology of this simplicial complex.



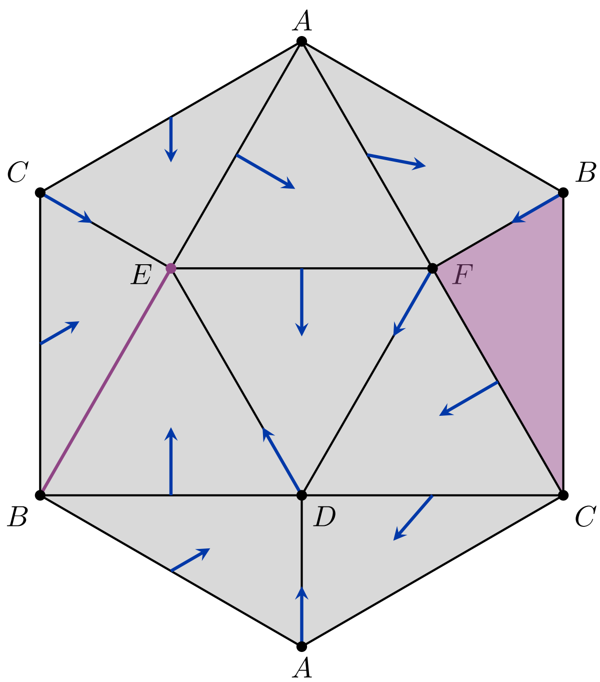
PROBLEM 59.

Use the given discrete vector field on the cylinder to compute its homology.



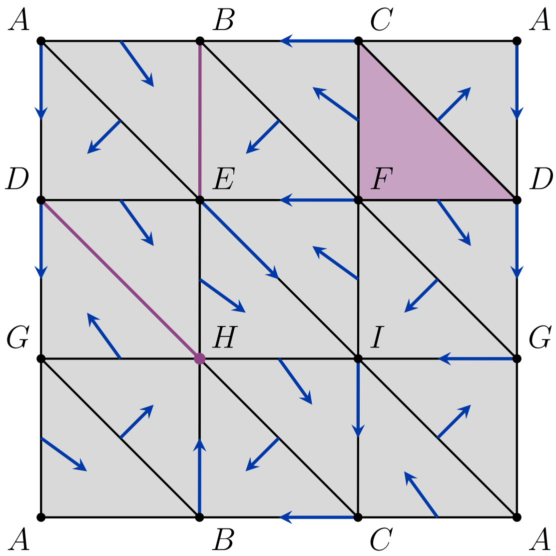
PROBLEM 60.

Use the given discrete vector field on the projective plane $\mathbb{R}P^2$ to compute its homology.



Problem 61.

Use the given discrete vector field on the torus to compute its homology.



Solutions

1. Metric spaces and homeomorphisms

Solution 1.

a. The powerset of X is

$$\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}.$$

Any topology $\mathcal{T}$ on X will have to contain $\emptyset$ and $\{0,1\}$, so we only need to decide if it should contain the singletons $\{0\}$ and $\{1\}$. This gives us four candidates for topologies on X:

$$\begin{aligned}
\mathcal{T}_1 &= \{\emptyset, \{0,1\}\} \text{ (trivial topology)},\\
\mathcal{T}_2 &= \{\emptyset, \{0\}, \{0,1\}\},\\
\mathcal{T}_3 &= \{\emptyset, \{1\}, \{0,1\}\},\\
\mathcal{T}_4 &= \{\emptyset, \{0\}, \{1\}, \{0,1\}\} \text{ (discrete topology)}.
\end{aligned}$$

We do not need to check the union and intersection conditions for $\emptyset$ and $\{0,1\}$, so we only need to verify that $\{0\} \cup \{1\} = \{0,1\} \in \mathcal{T}_4$ and $\{0\} \cap \{1\} = \emptyset \in \mathcal{T}_4$ to see that all four candidates listed above are indeed topologies on X.

b. The powerset of Y is

$$\mathcal{P}(Y) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Any topology $\mathcal{T}$ on X will have to contain $\emptyset$ and $\{0,1,2\}$, but in this case we have to decide if our topology will contain any of the remaining $2^3 - 2 = 6$ sets. This would give us $2^6 = 64$ candidates to check for the union and intersection condition. We can lower the number of cases that we need to check by observing the following. If a topology contains all three singletons $\{0\}$, $\{1\}$ and $\{2\}$, then the union rule implies it contains all possible subsets and is therefore equal to the discrete topology. **Any non-discrete topology can contain only 0, 1 or 2 singletons.** If a topology contains all three two-element sets $\{0,1\}$, $\{0,2\}$ and $\{1,2\}$, the intersection rule implies it must also contain all three singletons, and it is therefore equal to the discrete topology. **Any non-discrete topology can contain only 0, 1 or 2 two-element sets.** If a topology contains two two-element sets, it must also contain the singleton that is their intersection.

Any topology that is not trivial or discrete can thus contain

- 0 singletons and 1 two-element set or
- 1 singleton and 0 two-element set or
- 1 singleton and 1 two-element set or
- 1 singleton and 2 two-element set or
- 2 singletons and 1 two-element set or
- 2 singletons and 2 two-element set.

If we ignore the names of the points and only list topologies that differ in the number of singletons, two-element sets or both, we get 9 distinct topologies on Y. Allowing for all possible namings of points we obtain 29 topologies on Y.

$$\begin{aligned}
\mathcal{T}_1 &= \{\emptyset, \{0,1,2\}\} \text{ (trivial topology, 1)},\\
\mathcal{T}_2 &= \{\emptyset, \{0,1\}, \{0,1,2\}\} \text{ (3)},\\
\mathcal{T}_3 &= \{\emptyset, \{0\}, \{0,1,2\}\} \text{ (3)},\\
\mathcal{T}_4 &= \{\emptyset, \{0\}, \{1,2\}, \{0,1,2\}\} \text{ (3)},\\
\mathcal{T}_5 &= \{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}\} \text{ (6)},\\
\mathcal{T}_6 &= \{\emptyset, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\} \text{ (3)},\\
\mathcal{T}_7 &= \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,1,2\}\} \text{ (3)},\\
\mathcal{T}_8 &= \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}, \{0,1,2\}\} \text{ (6)},\\
\mathcal{T}_9 &= \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\} \text{ (discrete topology, 1)}.
\end{aligned}$$

There are 355 distinct topologies on $X = \{0,1,2,3\}$, or 33 if we ignore the names of the points. Can you write an efficient algorithm to find them all? It would be helpful to know that topologies on a finite set X are in one-to-one correspondence with preorders on X. A preorder on X is a binary relation $R \subset X \times X$, which is reflexive and transitive. So,

- $(x,x) \in R$ for all $x \in X$ and
- if $(x,y), (y,z) \in R$, then $(x,z) \in R$ for all $x,y,z \in X$.

There are $4 \cdot 4 = 16$ elements in $X \times X$, and the diagonal 4 must be included. That leaves us with $2^{12} = 4096$ candidates for preorders. On the other hand, $\mathcal{P}X$ also has $2^4 = 16$ elements (14 other than $\emptyset$ and X), which would give us $2^{14} = 16384$ candidates for topologies on X. If $|X| = n$, then there are 2^{n^2-n} preorder candidates and 2^{2^n-2} topology candidates. Preorders quickly become the more efficient way of looking for topologies although both approaches are computationally intensive.

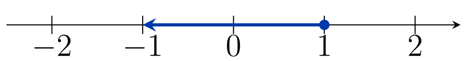
n	2	3	4	5	6	7	8	9
$n^2 - n$	2	6	12	20	30	42	56	72
$2^n - 2$	2	6	14	30	62	126	254	510

The number of different topologies on a set with n labeled elements belongs to the A000798 sequence in Sloane's On-line Encyclopedia of Integer Sequences. As of 2007 the terms of the sequence are only known up to $n = 18$. Similarly, the number of different topologies on a set with n unlabeled elements belongs to A001930. As of 2006 the terms of the sequence are only known up to $n = 16$.

Solution 2.

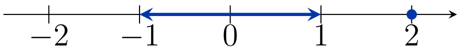
Using the definition of the open ball, we get

$$\begin{aligned}
B(1,2) &= \{x \in \mathbb{R} \,;\, d(x,1) < 2\} = \\
&= \{x \in \mathbb{R} \,;\, x = 1 \text{ and } 0 < 2\} \cup \{x \in \mathbb{R} \,;\, x \neq 1 \text{ and } |x| + |1| < 2\} = \\
&= \{1\} \cup \{x \in \mathbb{R} \,;\, |x| < 1\}.
\end{aligned}$$



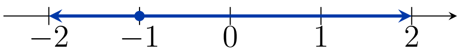
Similarly, we have

$$\begin{aligned} B(2,3) &= \{x \in \mathbb{R}\ ;\ d(x,2) < 3\} = \\ &= \{x \in \mathbb{R}\ ;\ x = 2 \text{ and } 0 < 3\} \cup \{x \in \mathbb{R}\ ;\ x \neq 2 \text{ and } |x| + |2| < 3\} = \\ &= \{2\} \cup \{x \in \mathbb{R}\ ;\ |x| < 1\}. \end{aligned}$$



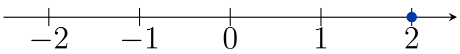
Notice how in both cases the centre of the ball is contained in the ball. Can you prove that this is always the case?
Next, we have

$$\begin{aligned} B(-1,3) &= \{x \in \mathbb{R}\ ;\ d(x,-1) < 3\} = \\ &= \{x \in \mathbb{R}\ ;\ x = -1 \text{ and } 0 < 3\} \cup \{x \in \mathbb{R}\ ;\ x \neq -1 \text{ and } |x| + |-1| < 3\} = \\ &= \{-1\} \cup \{x \in \mathbb{R}\ ;\ |x| < 2\}. \end{aligned}$$



Finally,

$$\begin{aligned} B(2,1) &= \{x \in \mathbb{R}\ ;\ d(x,2) < 1\} = \\ &= \{x \in \mathbb{R}\ ;\ x = 2 \text{ and } 0 < 1\} \cup \{x \in \mathbb{R}\ ;\ x \neq 2 \text{ and } |x| + |2| < 1\} = \\ &= \{2\} \cup \{\} = \\ &= \{2\}. \end{aligned}$$



Solution 3.

Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then

$$d_1(x,y) = \|x - y\|_1 = |x_1 - y_1| + |x_2 - y_2|,$$

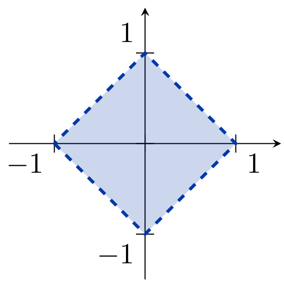
$$d_2(x,y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

and

$$d_\infty(x,y) = \|x - y\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

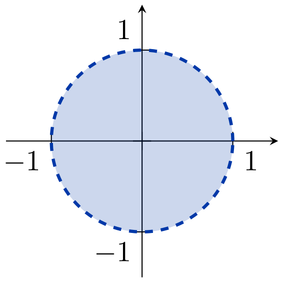
For $p = 1$ we get

$$B_1(0,1) = \{x \in \mathbb{R}^2\ ;\ d_1(x,0) < 1\} = \{(x_1,x_2) \in \mathbb{R}^2\ ;\ |x_1| + |x_2| < 1\}.$$



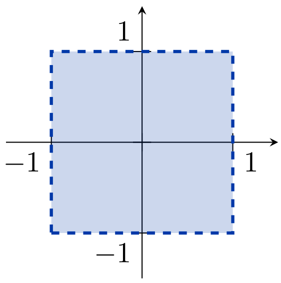
For $p = 2$ we get the standard unit ball in $\mathbb{R}^2$:

$$B_2(0,1) = \{x \in \mathbb{R}^2\ ;\ d_2(x,0) < 1\} = \{(x_1,x_2) \in \mathbb{R}^2\ ;\ \sqrt{x_1^2 + x_2^2} < 1\}.$$

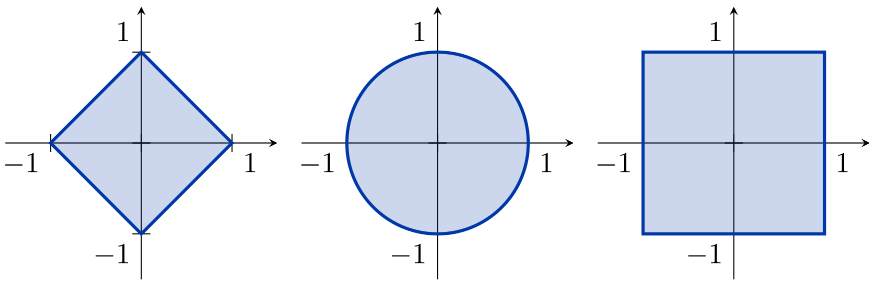


Finally, for $p = \infty$ we have

$$B_\infty(0,1) = \{x \in \mathbb{R}^2\ ;\ d_\infty(x,0) < 1\} = \{(x_1,x_2) \in \mathbb{R}^2\ ;\ \max\{|x_1|,|x_2|\} < 1\}.$$



To get $\overline{B}(0,1)$ in all three cases we simply need to replace $<$ by $\leq$, which means we must also include the boundary.



How would you draw $B(x_0,r)$ and $\overline{B}(x_0,r)$ for an arbitrary centre $x_0 \in \mathbb{R}^2$ and radius $r > 0$?

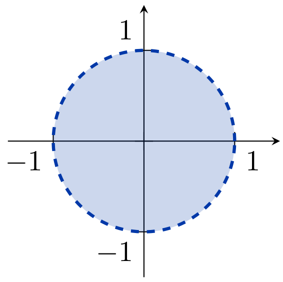
Solution 4.

a. With this metric we have two possible cases depending on whether or not the line through the two points also passes through the origin of the coordinate system. By definition

$$B((0,0),1) = \{(x_1,x_2) \in \mathbb{R}^2\ ;\ d((0,0),(x_1,x_2)) < 1\}.$$

Since the line through $(0,0)$ and (x_1,x_2) obviously contains the origin, we only need to look at the first case. We get

$$\begin{aligned} B((0,0),1) &= \{(x_1,x_2) \in \mathbb{R}^2\ ;\ \|(x_1,x_2)\|_2 < 1\} = \\ &= \{(x_1,x_2) \in \mathbb{R}^2\ ;\ \sqrt{x_1^2 + x_2^2} < 1\} = \\ &= \{(x_1,x_2) \in \mathbb{R}^2\ ;\ x_1^2 + x_2^2 < 1\}. \end{aligned}$$



b. The line through (x_1, x_2) and $(3, 0)$ will contain the origin if and only if $x_2 = 0$. So

$$d((3,0),(x_1,x_2)) = \begin{cases} \|(x_1,0)-(3,0)\|_2, & x_2 = 0, \\ \|(x_1,x_2)\|_2 + \|(3,0)\|_2, & x_2 \neq 0. \end{cases}$$

Since

$$\|(x_1,0)-(3,0)\|_2 = \|(x_1-3,0)\|_2 = \sqrt{(x_1-3)^2+0^2} = |x_1-3|$$

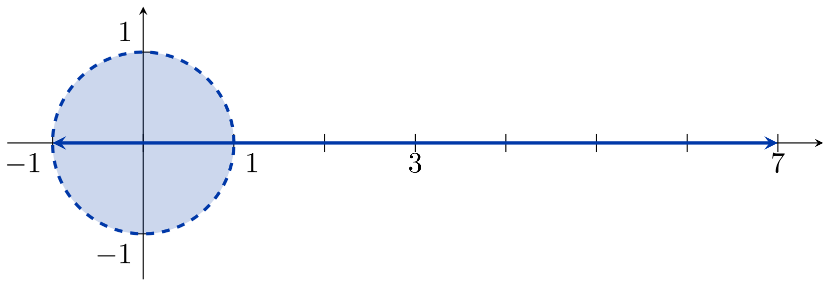
and

$$\|(x_1,x_2)\|_2 + \|(3,0)\|_2 = \sqrt{x_1^2+x_2^2}+3,$$

we get

$$\begin{aligned} B((3,0),4) &= \{(x_1,x_2)\in\mathbb{R}^2 \,;\, d((3,0),(x_1,x_2)) < 4\} = \\ &= \{(x_1,0)\,;\, |x_1-3|<4\} \cup \{(x_1,x_2)\,;\, \sqrt{x_1^2+x_2^2}+3<4,\ x_2\neq 0\} = \\ &= \{(x_1,0)\,;\, |x_1-3|<4\} \cup \{(x_1,x_2)\,;\, x_1^2+x_2^2<1,\ x_2\neq 0\}. \end{aligned}$$

The first set contains all points on the x-axis that are at most 4 away from 3. In other words, it is the open interval $(-1, 7)$. The second set is of course the standard unit open ball.



c. The line through (x_1, x_2) and $(1, 1)$ will contain the origin if and only if $x_1 = x_2$. So

$$d((1,1),(x_1,x_2)) = \begin{cases} \|(x_1,x_1)-(1,1)\|_2, & x_1 = x_2, \\ \|(x_1,x_2)\|_2 + \|(1,1)\|_2, & x_1 \neq x_2. \end{cases}$$

Since

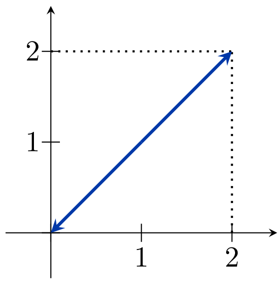
$$\|(x_1,x_1)-(1,1)\|_2 = \|(x_1-1,x_1-1)\|_2 = \sqrt{2(x_1-1)^2} = |x_1-1|\cdot\sqrt{2}$$

and

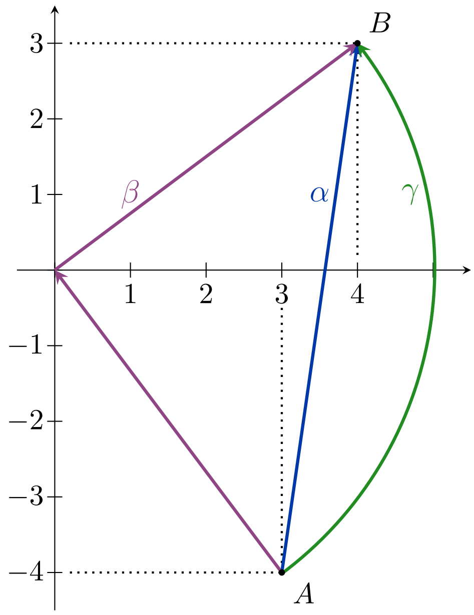
$$\|(x_1,x_2)\|_2 + \|(1,1)\|_2 = \sqrt{x_1^2+x_2^2}+\sqrt{2},$$

we get

$$\begin{aligned} B((1,1),\sqrt{2}) &= \{(x_1,x_2)\in\mathbb{R}^2 \,;\, d((1,1),(x_1,x_2)) < \sqrt{2}\} = \\ &= \{(x_1,x_1)\,;\, |x_1-1|<1\} \cup \{(x_1,x_2)\,;\, x_1^2+x_2^2<0,\ x_1\neq x_2\} = \\ &= \{(x_1,x_1)\,;\, |x_1-1|<1\} \cup \{\} = \\ &= \{(x_1,x_1)\,;\, |x_1-1|<1\}. \end{aligned}$$



SOLUTION 5.



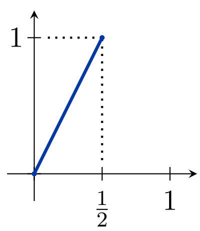
Let α be the path from A to B along the straight line segment. The line segment AB can be parametrized as

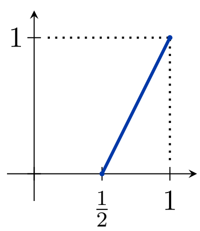
$$(1-t)A + tB, \ t \in [0,1],$$

so

$$\alpha(t) = (1-t)(3,-4) + t(4,3) = (3-3t, -4+4t) + (4t, 3t) = (t+3, 7t-4).$$

Let β be the path that takes a detour through the origin of the coordinate system. Here as well we choose to move along the straight line segments AO and OB. We use the first half of $[0,1]$ to move from A to O and the second half of $[0,1]$ to move from O to B. We begin by finding the reparametrizations $[0, \frac{1}{2}] \to [0,1]$ and $[\frac{1}{2}, 1] \to [0,1]$.





The first one corresponds to the straight line with slope 2 that passes through the origin, so $t \mapsto 2t$. The second also has slope 2 but passes through $(0,-1)$, so $t \mapsto 2t-1$.

Let β_1 be the path from A to O. As before, we can find its parametrization for $t \in [0,1]$ as

$$\beta_1(t) = (1-t)A + tO = (1-t)(3,-4) + t(0,0) = (3-3t, -4+4t).$$

Combining this with the first reparametrization $t \mapsto 2t$, we get

$$\beta_1'(t) = (3 - 3 \cdot 2t, -4 + 4 \cdot 2t) = (3 - 6t, 8t - 4),$$

a parametrization of the path that goes from A to O for $t \in [0, \frac{1}{2}]$. Similarly, we get the path β_2 from O to B for $t \in [0, 1]$ as

$$\beta_2(t) = (1 - t)O + tB = (1 - t)(0, 0) + t(4, 3) = (4t, 3t).$$

Combining this with the second reparametrization $t \mapsto 2t - 1$, we get

$$\beta_2'(t) = (4(2t - 1), 3(2t - 1)) = (8t - 4, 6t - 3),$$

a parametrization of the path that goes from O to B for $t \in [\frac{1}{2}, 1]$. Our path β has to run along β_1' for the first half of $[0, 1]$ and along β_2' for the second half, so

$$\beta(t) = \begin{cases} (3 - 6t, 8t - 4), & 0 \leq t \leq \frac{1}{2}, \\ (8t - 4, 6t - 3), & \frac{1}{2} < t \leq 1. \end{cases}$$

Note that the two descriptions agree for $t = \frac{1}{2}$, so this path is indeed continuous.

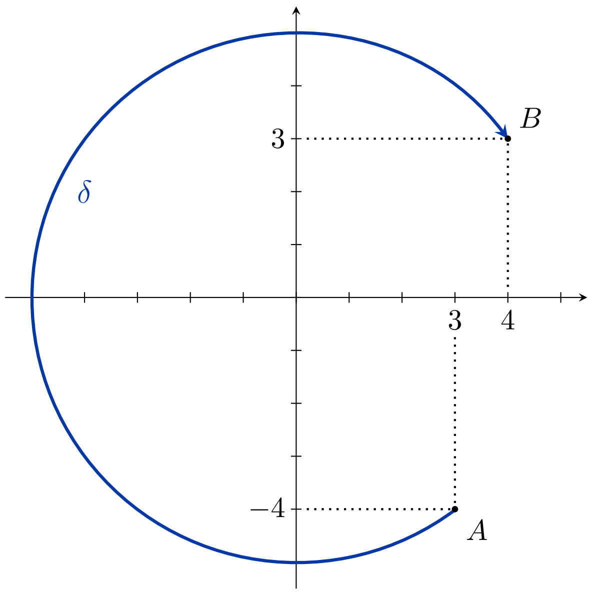
Finally, we notice that $d(A, O) = d(B, O) = 5$, so we can also get from A to B by moving along the circular arc of radius 5 centered at the origin. Let us call this path γ. The easiest way to obtain one possible parametrization of γ is to take the parametrization of α, normalize it to get the corresponding movement along the unit circle and multiply by 5 to move it to a circle of radius 5. Since

$$\|\alpha(t)\| = \|(t + 3, 7t - 4)\| = \sqrt{(t + 3)^2 + (7t - 4)^2} = \sqrt{50t^2 - 50t + 25} = 5\sqrt{2t^2 - 2t + 1},$$

we get

$$\gamma(t) = \frac{5\alpha(t)}{\|\alpha(t)\|} = \frac{(t + 3, 7t - 4)}{\sqrt{2t^2 - 2t + 1}} = \left(\frac{t + 3}{\sqrt{2t^2 - 2t + 1}}, \frac{7t - 4}{\sqrt{2t^2 - 2t + 1}} \right).$$

Try to use all the tricks from this problem to construct the path δ that moves from A to B along the circle of radius 5 centered at O, but takes the long way around the origin. This process of constructing such paths may seem a bit tedious. We will learn a much better way to do so in the next problem.



Solution 6.

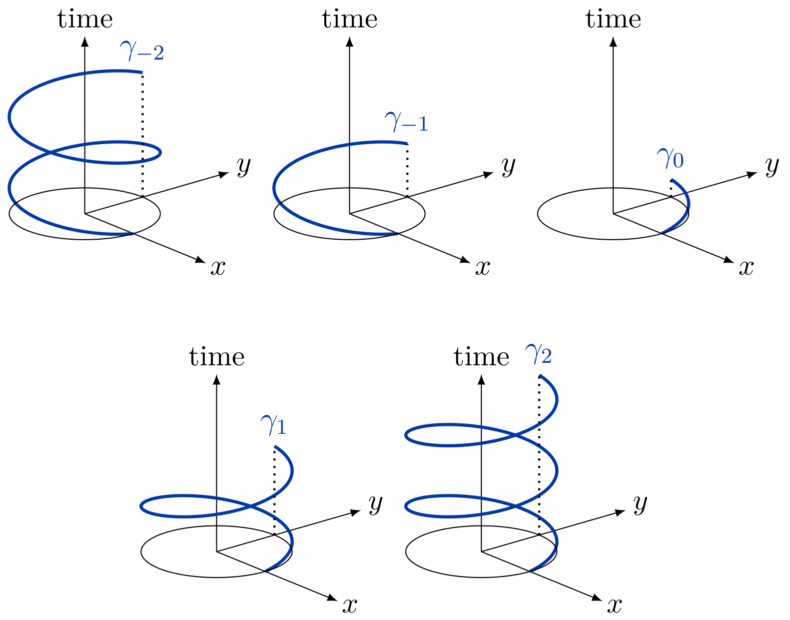
Each point in the plane can be written in the form $(r\cos\varphi, r\sin\varphi)$ or, equivalently, as $re^{i\varphi}$. If we restrict the radius to 1, we will stay on the unit circle. We need to get from $(1,0)$, which is represented as $e^{i\,0}$ to $(0,1)$ or $e^{i\frac{\pi}{2}}$. We need to vary the angle from 0 to $\frac{\pi}{2}$ while the parameter t goes from 0 to 1. In other words, $\varphi = \frac{\pi}{2}t$. This gives us the path

$$\gamma_0(t) = e^{i\frac{\pi}{2}t} = \left(\cos\left(\tfrac{\pi}{2}t\right), \sin\left(\tfrac{\pi}{2}t\right)\right).$$

We can also add any number of full turns to our path. The point $(0,1)$ can also be represented as $e^{i\left(\frac{\pi}{2}+2\pi k\right)}$ for any $k \in \mathbb{Z}$. If k is positive, we add some counterclockwise turns to γ_0 and for negative k we turn in the clockwise direction. In this case we need to vary the angle from 0 to $\frac{\pi}{2} + 2\pi k$ as t goes from 0 to 1, so $\varphi = \left(\frac{\pi}{2} + 2\pi k\right) t$. We get an infinite family of paths

$$\gamma_k(t) = e^{i\left(\frac{\pi}{2}+2\pi k\right)t} = \left(\cos\left(\tfrac{\pi}{2} + 2\pi k\right) t, \sin\left(\tfrac{\pi}{2} + 2\pi k\right) t\right).$$

The following images show the paths γ_k for $k \in \{-2,-1,0,1,2\}$. Time is plotted on a separate axis so that the paths do not intersect themselves. In reality each γ_k lies entirely in the xy-plane. The scale on the time axis is also different from image to image in order to make sure all the details are visible. The endpoint of the path is always where $t = 1$.

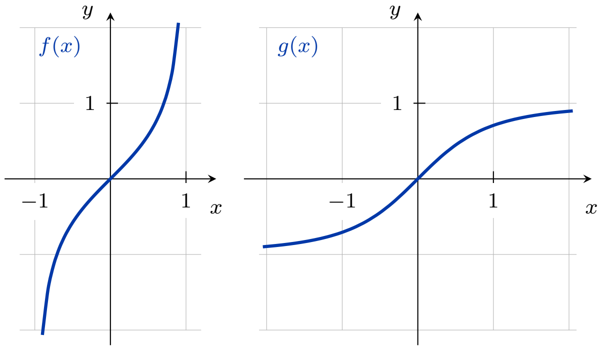


We see that γ_k winds roughly k times around the origin (slightly less than that because the starting and ending points are not the same). We could also define closed paths α_k from $(1,0)$ to $(1,0)$ that wind around the origin exactly k times counterclockwise for positive k and clockwise for negative k:

$$\alpha_k(t) = e^{i(2\pi k)t} = (\cos 2\pi kt, \sin 2\pi kt).$$

In this case k is the winding number of the path α_k around the origin. Winding number is defined for any closed path α in the plane around any point p in the plane that does not lie on the path α itself.

Solution 7.

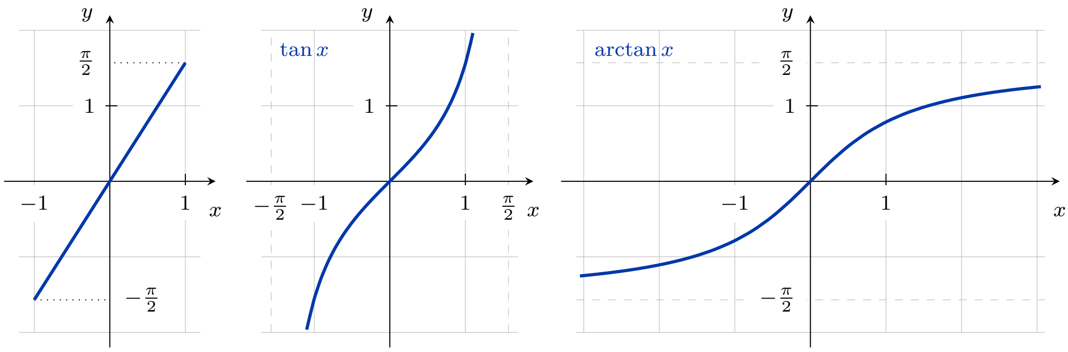


We have

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) = f\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} = \\
&= \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1-\frac{x^2}{1+x^2}}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{\frac{1+x^2-x^2}{1+x^2}}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\frac{1}{\sqrt{1+x^2}}} = x.
\end{aligned}
$$

so $f \circ g = \mathrm{id}_{\mathbb{R}}$. A very similar calculation shows that $g \circ f = \mathrm{id}_{(-1,1)}$. This shows that $(-1, 1)$ and $\mathbb{R}$ are homeomorphic.

The graph of f might remind you of the function $\tan x$. In fact, we can use $f_1(x) = \tan x$ and its inverse $g_1(x) = \arctan x$ to show that $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$ are homeomorphic. To get a homeomorphism between $(-1, 1)$ and $\mathbb{R}$ we need to precompose f with any bijection $(-1, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$, for example $x \mapsto \frac{\pi}{2}x$. We get a map $f_2\colon (-1, 1) \to \mathbb{R}$, $f_2(x) = \tan(\frac{\pi}{2}x)$ which is also a homeomorphism between $(-1, 1)$ and $\mathbb{R}$. Its inverse is $g_2(x) = \frac{2}{\pi}\arctan x$.



To show that the open interval $(-1, 1)$ and the closed interval $[-1, 1]$ are not homeomorphic assume there exists a homeomorphism $f\colon [-1, 1] \to (-1, 1)$. Now, the space $X = [-1, 1] - \{1\} = [-1, 1)$ is connected and the space $Y = (-1, 1) - \{f(1)\} = (-1, f(1)] \cup [f(1), 1)$ is not connected, so they are not homeomorphic. But restricting the homeomorphism f to X should give us a homeomorphism between X and Y, a contradiction. We conclude that such homeomorphism f does not exist.

Are the open interval $(0, 1)$ and the half-open interval $(0, 1]$ homeomorphic? Are the half-open interval $(0, 1]$ and the closed interval $[0, 1]$ homeomorphic? How about $(a, b]$ and $[c, d)$? Is $(0, 1]$ homeomorphic to $\mathbb{R}$? Is $[0, 1]$ homeomorphic to $\mathbb{R}$? How would you prove it?

Solution 8.

We can assume that $n = 1$ without loss of generality. The calculation for $n > 1$ is exactly the same, whatever happens with the first component when $n = 1$ happens to the first n components in the general case.

In case of $n=1$ we have $X = S^1 \setminus \{(0,1)\} = \{(x,y) \in \mathbb{R}^2 \; ; \; x^2+y^2=1\}$ and $Y=\mathbb{R}$. The two maps are $f(x,y) = \frac{x}{1-y}$ and

$$g(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right).$$

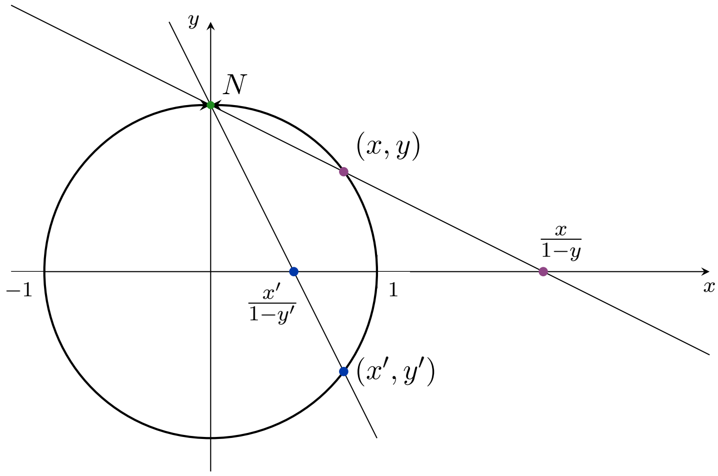
We can check that

$$\begin{aligned}(f\circ g)(x) &= f(g(x)) = f\left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right) = \frac{\frac{2x}{x^2+1}}{1-\frac{x^2-1}{x^2+1}} = \\ &= \frac{\frac{2x}{x^2+1}}{\frac{x^2+1-x^2+1}{x^2+1}} = \frac{2x}{2} = x\end{aligned}$$

and

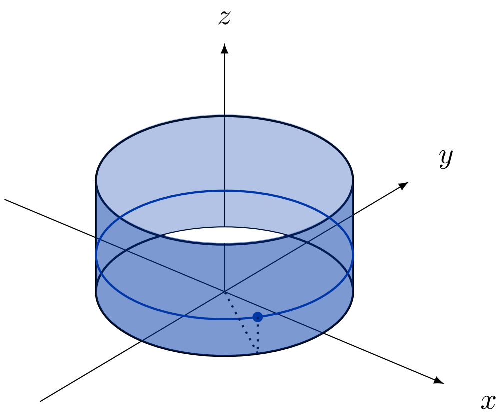
$$\begin{aligned}(g\circ f)(x,y) &= g(f(x,y)) = g\left(\frac{x}{1-y}\right) = \left(\frac{2\frac{x}{1-y}}{\left(\frac{x}{1-y}\right)^2+1}, \frac{\left(\frac{x}{1-y}\right)^2-1}{\left(\frac{x}{1-y}\right)^2+1} \right) = \\ &= \left(\frac{\frac{2x}{1-y}}{\frac{x^2}{1-2y+y^2}+1}, \frac{\frac{x^2}{1-2y+y^2}-1}{\frac{x^2}{1-2y+y^2}+1} \right) = \left(\frac{\frac{2x}{1-y}}{\frac{x^2+1-2y+y^2}{1-2y+y^2}}, \frac{\frac{x^2-1+2y-y^2}{1-2y+y^2}}{\frac{x^2+1-2y+y^2}{1-2y+y^2}} \right) = \\ &= \left(\frac{\frac{2x}{1-y}}{\frac{1+1-2y}{(1-y)^2}}, \frac{x^2-1+2y-y^2}{1+1-2y} \right) = \left(\frac{2x(1-y)}{2-2y}, \frac{x^2-1+2y-y^2}{2-2y} \right) = \\ &= \left(\frac{2x(1-y)}{2(1-y)}, \frac{-y^2+2y-y^2}{2(1-y)} \right) = \left(x, \frac{2y(1-y)}{2(1-y)} \right) = (x,y).\end{aligned}$$

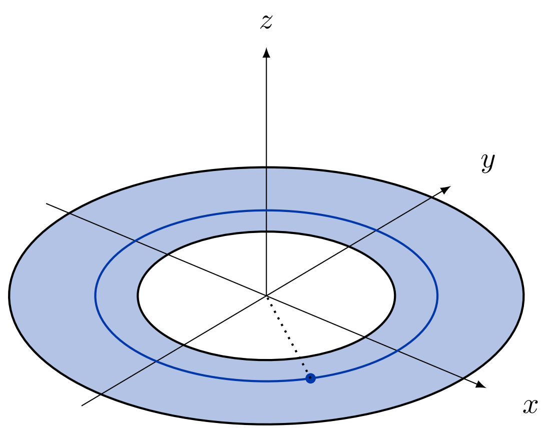
We see that $f\circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, so the two maps are homeomorphisms between X and Y. The map f is called the **stereographic projection**.



The points on $S^1 \setminus \{(0,1)\}$ get projected to $\mathbb{R}$ by extending the line through the point and the north pole until it meets the x-axis. The points on the upper half of the circle get projected to the complement of $(-1,1)$, the points in the lower hemisphere end up on $(-1,1)$. The points $(1,0)$ and $(-1,0)$ map to 1 and -1, respectively.

Solution 9.





Let us write

$$X = \{(x, y, z) \in \mathbb{R}^3 \ ; \ x^2 + y^2 = 1, \ 0 \le z \le 1\}$$

and recall that

$$Y = \{(x, y) \in \mathbb{R}^2 \ ; \ 0 \le x^2 + y^2 \le 4\}.$$

We need to find continuous maps $f\colon X \to Y$ and $g\colon Y \to X$, such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. There are many possible solutions because both figures are invariant with respect to rotations around the z-axis, but perhaps the simplest way to map a point from X to Y is to leave the angle between the x-axis and the line segment connecting the projection of the point to the xy-plane to the origin unchanged and simply scale the length of this segment depending on the z-coordinate. The map from Y to X then needs to reverse this process exactly. The scaling factor r will depend on the z-coordinate. Points in the xy-plane with $z = 0$ can remain on the circle of radius $r = 1$. Points at the top of the cylinder with $z = 1$ need to get mapped to the circle of radius $r = 2$ (to the outside edge of the annulus). For z in between 0 and 1 we can interpolate using a simple linear function, so $r = z + 1$. The angle will be preserved if the ratio between the x and y coordinates remains the same. Hence,

$$f(x, y, z) = ((z + 1)x, (z + 1)y).$$

We need to verify that $((z + 1)x, (z + 1)y) \in Y$. To that end we calculate

$$((z + 1)x)^2 + ((z + 1)y)^2 = (z + 1)^2(x^2 + y^2) = (z + 1)^2.$$

We have used the fact that $x^2 + y^2 = 1$ because $(x, y, z) \in X$. From $0 \le z \le 1$ we get $1 \le (z + 1)^2 \le 4$, so the condition of Y is indeed satisfied.

To get g, we need to use the distance from the origin $r = \sqrt{x^2 + y^2}$ of the point (x, y) to determine the height $z = r - 1 = \sqrt{x^2 + y^2} - 1$. We also need to scale the first two coordinates to make sure we end up over the unit circle. So,

$$g(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} - 1 \right).$$

To verify that we have landed in X, we need to check that

$$\left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2 = \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1$$

and that $1 \le x^2 + y^2 \le 4$ implies that $0 \le z = \sqrt{x^2 + y^2} - 1 \le 1$.

Finally, we need to compute $f \circ g$ and $g \circ f$ to show that the two maps are inverses of one another. We have

$$\begin{aligned}
(f \circ g)(x,y) &= f\left(g(x,y)\right) = \\
&= f\left(\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \sqrt{x^2+y^2}-1\right)\right) = \\
&= \left(\left(\sqrt{x^2+y^2}-1+1\right)\cdot\frac{x}{\sqrt{x^2+y^2}}, \left(\sqrt{x^2+y^2}-1+1\right)\cdot\frac{y}{\sqrt{x^2+y^2}}\right) = \\
&= (x,y),
\end{aligned}$$

so $(f \circ g) = \mathrm{id}_Y$. For the other composition we will use the fact that

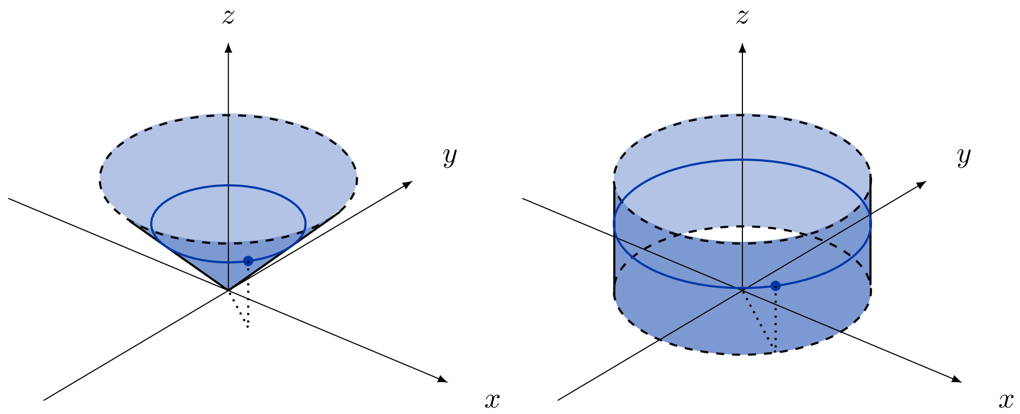
$$\sqrt{((z+1)x)^2+((z+1)y)^2} = \sqrt{(z+1)^2(x^2+y^2)} = \sqrt{(z+1)^2} = |z+1| = z+1.$$

We get

$$\begin{aligned}
(g \circ f)(x,y) &= g\left(f(x,y,z)\right) = \\
&= g\left(((z+1)x, (z+1)y)\right) = \\
&= \left(\frac{(z+1)x}{z+1}, \frac{(z+1)y}{z+1}, z+1-1\right) = \\
&= (x,y,z),
\end{aligned}$$

so $(g \circ f) = \mathrm{id}_X$.

SOLUTION 10.



Recall that

$$X = \{(x,y,z) \in \mathbb{R}^3\ ;\ z^2 = x^2+y^2,\ 0<z<1\}$$

and write

$$Y = \{(x,y,z) \in \mathbb{R}^3\ ;\ x^2+y^2 = 1,\ 0<z<1\}.$$

For any fixed $z \in (0,1)$ the points in X lie on the circle of radius z at height z and the points in Y lie on the circle of radius 1 at the same height z, so all we need to do is to scale the radius by the factor z to get $f\colon X \to Y$,

$$f(x,y,z) = \left(\frac{x}{z}, \frac{y}{z}, z\right).$$

The inverse map $g\colon Y \to X$ is

$$g(x,y,z) = (zx, zy, z).$$

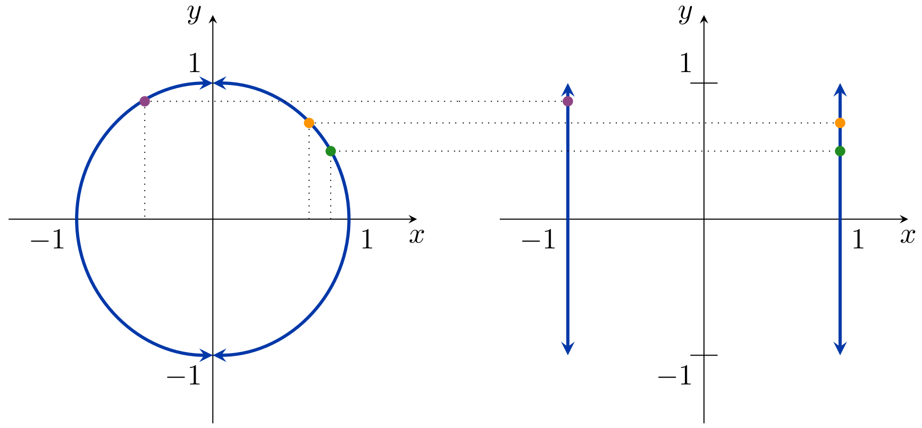
We see that

$$(f \circ g)(x,y,z) = f(g(x,y,z)) = f(zx,zy,z) = \left(\frac{zx}{z}, \frac{zy}{z}, z\right) = (x,y,z)$$

and
$$(g\circ f)(x,y,z)=g(f(x,y,z))=g\left(\frac{x}{z},\frac{y}{z},z\right)=\left(\frac{zx}{z},\frac{zy}{z},z\right)=(x,y,z).$$

Solution 11.

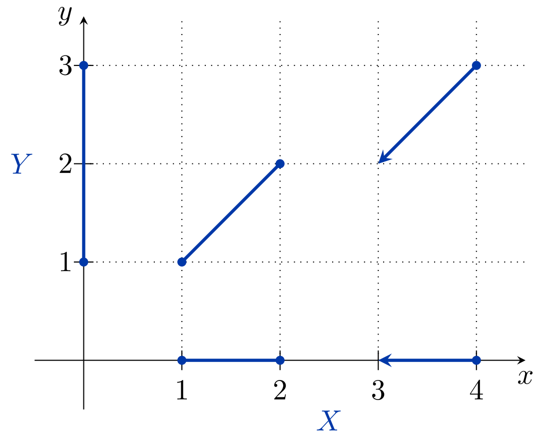
Let us begin by drawing X and Y for $n=1$. In this case we have
$$X_1=S^1\setminus\{(0,1),(0,-1)\}\subseteq\mathbb{R}^2$$
and
$$Y_1=S^0\times(-1,1)=\{-1,1\}\times(-1,1)\subseteq\mathbb{R}^2.$$



We see that we can open up the two circular arcs by setting the x-coordinate of each point to -1 or 1 as appropriate and keeping the y-coordinate constant, so the map $f_1\colon X_1\to Y_1$ can be defined as
$$f_1(x,y)=\begin{cases}(-1,y), & x<0,\\ (1,y), & x>0.\end{cases}$$
The inverse map g_1 needs to change the x coordinate from ±1 in such a way that the resulting point will lie on the circle. So, $g_1\colon Y_1\to X_1$ is
$$g_1(x,y)=(x\sqrt{1-y^2},y^2).$$
It is easy to check that $f_1\circ g_1=\mathrm{id}_{Y_1}$ and $g_1\circ f_1=\mathrm{id}_{X_1}$.

Now, we need to generalize these maps to maps $f_n\colon X_n\to Y_n$ and $g_n\colon Y_n\to X_n$, where
$$X_n=\{(x_1,\dots,x_n,x_{n+1})\in\mathbb{R}^{n+1}\ ;\ x_1^2+\ldots+x_n^2+x_{n+1}^2=1,x_{n+1}\neq\pm1\}$$
and
$$Y_n=\{(x_1,\dots,x_n,x_{n+1})\in\mathbb{R}^{n+1}\ ;\ x_1^2+\ldots+x_n^2=1,-1<x_{n+1}<1\}.$$
In order to make this easier, let us rewrite f_1 as
$$f_1(x,y)=\left(\frac{x}{\sqrt{1-y^2}},\,y\right).$$
It is not difficult to check that this is indeed the same map as before using the fact that $x^2=1-y^2$. We can now define
$$f_n(x_1,\dots,x_n,x_{n+1})=\left(\frac{x_1}{\sqrt{1-x_{n+1}^2}},\frac{x_2}{\sqrt{1-x_{n+1}^2}},\dots,\frac{x_n}{\sqrt{1-x_{n+1}^2}},x_{n+1}\right)$$
and
$$g_n(x_1,\dots,x_n,x_{n+1})=\left(x_1\sqrt{1-x_{n+1}^2},\dots,x_n\sqrt{1-x_{n+1}^2},x_{n+1}\right).$$
Once again it is not to difficult to check that $f_n\circ g_n=\mathrm{id}_{Y_n}$ and $g_n\circ f_n=\mathrm{id}_{X_n}$.

SOLUTION 12.

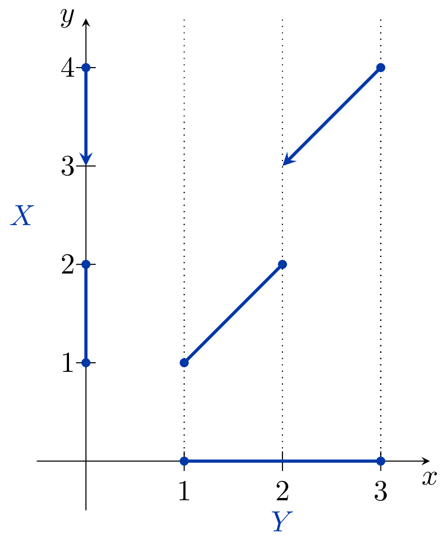


We can define $f\colon X \to Y$ as

$$f(x) = \begin{cases} x, & 1 \leq x \leq 2, \\ x-1, & 3 < x \leq 4. \end{cases}$$

Intuitively, f is continuous, because it brings the two intervals from X closer together. To be more precise, for any $x_0 \in X$ and $\varepsilon > 0$ we can find $\delta > 0$, such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. Indeed, we can just use $\delta = \varepsilon$ for all x_0.

The map f is also injective (if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in X$) and surjective (for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$), so it is a bijection.



The inverse $f^{-1}\colon Y \to X$ is

$$f^{-1}(y) = \begin{cases} y, & 1 \leq y \leq 2, \\ y+1, & 2 < y \leq 3. \end{cases}$$

The inverse is not continuous. For example, for $x_0 = 2$ and $\varepsilon = \frac{1}{2}$, there will always be points arbitrarily close to 2 whose funcion values will be at least 1 away, so no $\delta > 0$ exists.

This does not yet imply, however, that X and Y are not homeomorphic. All we have done is found one function that is not a homeomorphism. There are still infinitely many candidates to check. To show that X and Y are not homeomorphic, we need to find a topological property, a property that we know is preserved under homeomorphisms, and show that one of the two spaces has that property while the other one does not. Then we will be able to conclude that X and Y are not homeomorphic.

One of the topological properties that is useful in this case is path-connectedness. The space Y is path-connected: given any two points $y_1, y_2 \in Y$ we can always find a continuous path that starts at y_1 and ends at y_2. The space X is not path-connected: there is no continuous path between $x_1 = 1$ and $x_2 = 4$, for example.

Another possibility would be connectedness. Intuitively, Y has one component while X is made up of two pieces, so they are not homeomorphic.

Notice that the Euler characteristic (more on it later, just try to guess what it is for now) does not distinguish between X and Y. The former has three vertices and two edges, so $\chi(X) = 3 - 2 = 1$. The latter has two vertices and one edge, so once again $\chi(Y) = 2 - 1 = 1$. This is fine and does not imply that X and Y are homeomorphic, just that this particular invariant is not able to tell them apart. The goal with this type of problems (or maybe even the goal of topology in general) is to find the ones that do.

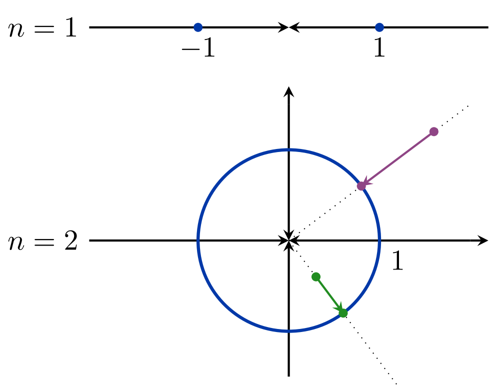
Solution 13.

None of the surfaces are homeomorphic. The cylinder and the Moebius strip are not homeomorphic to any of the other three or to each other because the cylinder has two boundary components, the Moebius strip has one and the rest have no boundary. The sphere is not homeomorphic to the torus or the Klein bottle because any loop on the sphere can be contracted to a point while this is not the case for the other two. The torus and the Klein bottle are not homeomorphic because the former is orientable and the latter is not.

The cylinder and the Moebius strip are homotopy equivalent because they are both homotopy equivalent to S^1 (this shows that orientability and dimension are not homotopy invariants). The sphere is not homotopy equivalent to any of the other four because any loop on the sphere can be contracted to a point while the rest all contain non-contractible loops. We will be able to distinguish the torus from the Klein bottle once we learn how to compute their homology.

Solution 14.

Let us first look at the cases $n = 0$ and $n = 1$ where we can easily see what is going on (try to draw the case $n = 2$ on your own). We can consider $S^0 = \{-1, 1\}$ (the unit 0-dimensional sphere) and S^1 (the unit circle) as subspaces of $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{(0,0)\}$, respectively.



The missing origin enables us to collapse $B^1 \setminus \{0\}$ and $B^2 \setminus \{(0,0)\}$ that are bounded by S^0 and S^1 to the boundary radially from the missing point. The ouside, on the other hand, can be collapsed inwards due to the "missing" point at infinity. The same trick will work in higher dimensions. To bring each point $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0^{n+1}\}$ onto S^n we only need to divide its every coordinate by the norm of the corresponding vector, so we can define $g\colon \mathbb{R}^{n+1} \setminus \{0^{n+1}\} \to S^n$,

$$g(x_1, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}}, \ldots, \frac{x_{n+1}}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}} \right).$$

The other map, $f\colon S^n \to \mathbb{R}^{n+1} \setminus \{0^{n+1}\}$, does not need to move the points at all since we already consider S^n as a subset of $\mathbb{R}^{n+1} \setminus \{0^{n+1}\}$. So, $f(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1})$. The composition $g \circ f\colon S^n \to S^n$ turns out to be very simple:

$$(g \circ f)(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}}, \ldots, \frac{x_{n+1}}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}} \right) = (x_1, \ldots, x_{n+1}).$$

The last equality holds because $\sqrt{x_1^2 + \ldots + x_{n+1}^2} = 1$ for the points in S^n. We see that $g \circ f = \mathrm{id}_{S^n}$, so we do not need to find the homotopy between them. On the other hand we have

$$(f \circ g)(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}}, \ldots, \frac{x_{n+1}}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}} \right),$$

so we need to find a homotopy

$$H\colon \mathbb{R}^{n+1} \setminus \{0^{n+1}\} \times I \to \mathbb{R}^{n+1} \setminus \{0^{n+1}\}$$

between the map $f \circ g\colon \mathbb{R}^{n+1} \setminus \{0^{n+1}\} \to \mathbb{R}^{n+1} \setminus \{0^{n+1}\}$ and $\mathrm{id}_{\mathbb{R}^{n+1} \setminus \{0^{n+1}\}}$. This homotopy will, in fact, describe the path that each point made as we imagined the surrounding space collapsing onto S^n. So, let

$$H(x_1, \ldots, x_{n+1}, t) = (1-t) \left(\frac{x_1}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}}, \ldots, \frac{x_{n+1}}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}} \right) + t(x_1, \ldots, x_{n+1}).$$

The denominator of the fractions $\sqrt{x_1^2 + \ldots + x_{n+1}^2}$ is never zero because we excluded the origin, so H is continuous. It is also obvious that

$$H(x_1, \ldots, x_{n+1}, 0) = \left(\frac{x_1}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}}, \ldots, \frac{x_{n+1}}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}} \right) = (f \circ g)(x_1, \ldots, x_{n+1})$$

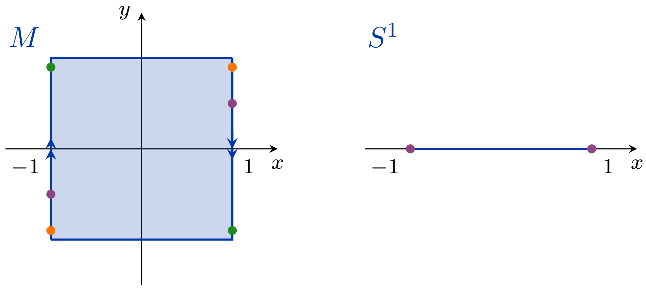
and

$$H(x_1, \ldots, x_{n+1}, 1) = (x_1, \ldots, x_{n+1}) = \mathrm{id}_{\mathbb{R}^{n+1} \setminus \{0^{n+1}\}}(x_1, \ldots, x_{n+1}),$$

as required.

Solution 15.

Let us begin by drawing the two spaces. The left and the right side of the square are glued together with a 180° twist to form the Moebius strip (some examples of points that coincide after the gluing process are shown in the same colour). The endpoints of the interval are glued together to form S^1.



To get a map $f\colon M \to S^1$ we just need to forget the second coordinate: $f(x, y) = x$. The map $g\colon S^1 \to M$ can include the circle into the Moebius band at level $y = 0$, identifying the point $g(x)$ with the point $(x, 0)$ in the plane. Note that trying to do this identification at any other level by identifying $g(x)$ with a point (x, y_0) for $y_0 \neq 0$ would not work because $(1, y_0)$ and $(-1, y_0)$ would not match up correctly to form a closed circle once the sides of the Moebius strip are glued with a twist.

Let us compute $f \circ g\colon S^1 \to S^1$ and $g \circ f\colon M \to M$:

$$\begin{array}{rcccccl} (f \circ g)(x) & = & f(g(x)) & = & f(x, 0) & = & x, \\ (g \circ f)(x, y) & = & g(f(x, y)) & = & f(x) & = & (x, 0). \end{array}$$

We see that $f \circ g = \mathrm{id}_{S^1}$. We only need to find a homotopy

$$H\colon M \times I \to M$$

between $g \circ f$ and id_M. Let

$$H(x, y, t) = (1 - t)(x, 0) + t(x, y) = (x - tx + tx, ty) = (x, ty).$$

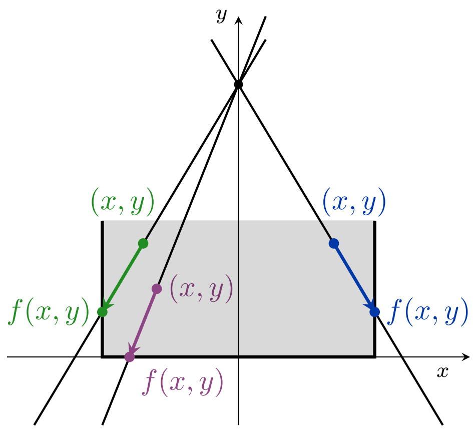
Then $H(x, y, 0) = (x, 0) = (g \circ f)(x, y)$ and $H(x, y, 1) = (x, y) = \mathrm{id}_M(x, y)$ for all $(x, y) \in M$. The map H is also clearly continuous, so this is the homotopy that shows $g \circ f \simeq \mathrm{id}_M$.

Solution 16.

The space X is a rectangle and Y consists of the south, east and west boundary segments. We can map X onto Y by projecting the points from any point in the region

$$\{(x, y) \in \mathbb{R}^2 \,;\, -1 < x < 1, y > 1\}.$$

Let us choose the point $(0, 2)$, for example.



The equation of the line through $(0, 2)$ and $(x_0, y_0) \in X$ is

$$y = \left(\frac{y_0 - 2}{x_0}\right) x + 2.$$

We see that

- the line passing through $(-1,1)$ is $y = x+2$,
- the line passing through $(-1,0)$ is $y = 2x+2$,
- the line passing through $(1,0)$ is $y = -2x+2$,
- the line passing through $(1,1)$ is $y = -x+2$.

If the slope $\frac{y_0-2}{x_0}$ is between

- -2 and $-\infty$, then the point (x_0, y_0) will be projected to the southern edge,
- -2 and -1, then the point (x_0, y_0) will be projected to the eastern edge,
- 1 and 2, then the point (x_0, y_0) will be projected to the western edge,
- 2 and ∞, then the point (x_0, y_0) will be projected to the southern edge.

Note that slopes between -1 and 1 are impossible to achieve with $(x_0, y_0) \in X$. A quick calculation shows that the lines $y = \left(\frac{y_0-2}{x_0}\right) x + 2$ intersect

- the southern edge (the line $y=0$) at $(-\frac{2x_0}{y_0-2}, 0)$,
- the eastern edge (the line $x=1$) at $(1, \frac{y_0-2}{x_0}+2)$ and
- the western edge (the line $x=-1$) at $(-1, -\frac{y_0-2}{x_0}+2)$.

We can therefore define $f\colon X \to Y$ as

$$f(x,y) = \begin{cases} (-\frac{2x_0}{y_0-2}, 0), & \frac{y_0-2}{x_0} < -2, \\ (1, \frac{y_0-2}{x_0}+2), & -2 \leq \frac{y_0-2}{x_0} \leq -1, \\ (-1, -\frac{y_0-2}{x_0}+2), & 1 \leq \frac{y_0-2}{x_0} \leq 2, \\ (-\frac{2x_0}{y_0-2}, 0), & 2 < \frac{y_0-2}{x_0}. \end{cases}$$

The map $g\colon Y \to X$ is slightly less complicated: $g(x,y) = (x,y)$.

Note that for all $(x,y) \in A$ we have $f(x,y) = (x,y)$ as well. This immediately implies that $f \circ g = \mathrm{id}_A$. To show that $g \circ f \simeq \mathrm{id}_X$ we define the homotopy $H\colon X \times I \to X$,

$$H(x,y,t) = (1-t)(g\circ f)(x,y) + t(x,y) = (1-t)f(x,y) + t(x,y).$$

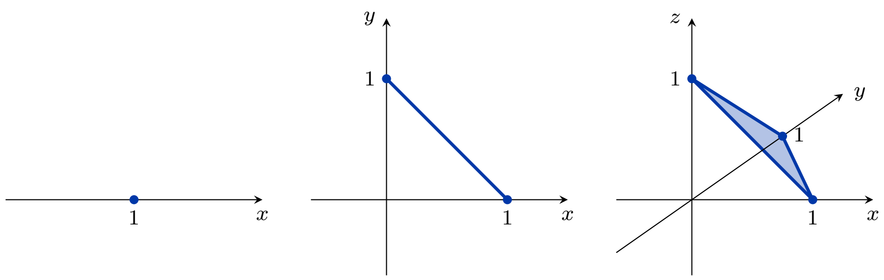
Because X is convex the line segments used by H remain inside X at all times, so H is well-defined. It is also clear from the definition that $H(x,y,0) = (g\circ f)(x,y)$ and $H(x,y,1) = \mathrm{id}_X(x,y)$, so we have shown that $X \simeq Y$.

Solution 17.

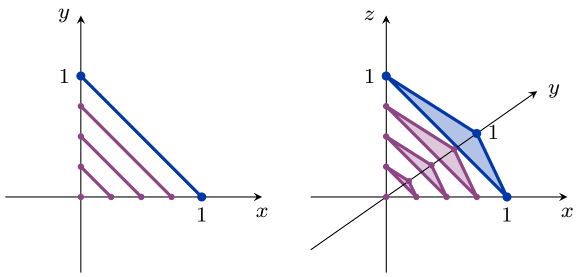
Let

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ ; \ x_1 + x_2 + \ldots + x_{n+1} = 1,\ x_i \geq 0 \text{ for all } i\}.$$

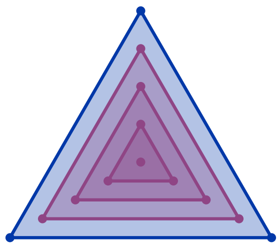
The following figure shows Δ^0, Δ^1 and Δ^2.



We may be tempted to contract Δ^n to the origin as shown below.



However, this would **not** show that Δ^n is contractible (as a space), only that it is contractible as a subspace of $\mathbb{R}^{n+1}$. This is not enough for the space itself to be contractible. For example, we can contract $S^1 \subseteq B^2$ to a point in B^2, but S^1 is not contractible (in S^1). Instead, we need to pick a point inside Δ^n, for example the centroid $\left(\frac{1}{n+1}, \dots, \frac{1}{n+1}, \frac{1}{n+1}\right)$, contract Δ^n to that point and make sure that the homotopy remains inside Δ^n at all times.



Define $H \colon \Delta^n \times I \to \Delta^n$,

$$H(x_1, \dots, x_{n+1}, t) = (1-t)(x_1, \dots, x_{n+1}) + t\left(\frac{1}{n+1}, \dots, \frac{1}{n+1}, \frac{1}{n+1}\right).$$

It is easy to see that

$$H(x_1, \dots, x_{n+1}, 0) = (x_1, \dots, x_{n+1}) = \mathrm{id}_{\Delta^n}(x_1, \dots, x_{n+1})$$

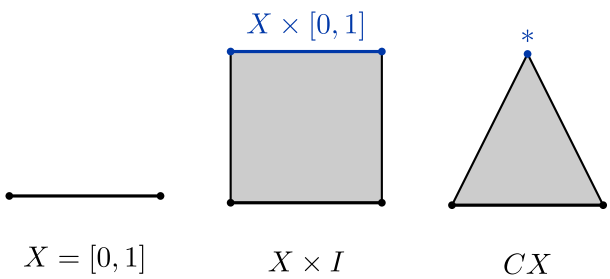
and

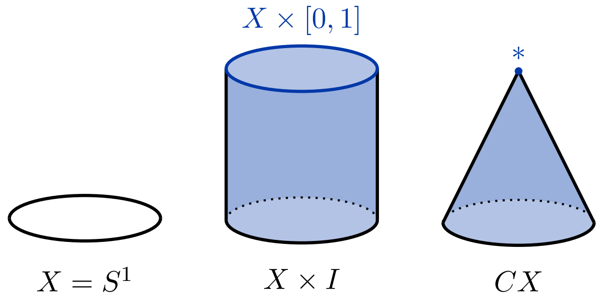
$$H(x_1, \dots, x_{n+1}, 1) = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1}, \frac{1}{n+1}\right) = \mathrm{c}(x_1, \dots, x_{n+1}),$$

so $H|_0$ is the identity map and $H|_1$ is the constant map $c \colon \Delta^n \to \Delta^n$ that maps the entire Δ^n to its centroid. The image of H remains inside Δ^n at all times, because we contract along the straight line segments and Δ^n is convex.

Solution 18.

For any space X we construct the cone CX by first constructing the product $X \times I$ and then collapsing the entire top level $X \times \{1\}$ to a single point we denote by $*$. Two examples are shown in the figure below.





The points of CX are of the form (x, s), where $x \in X$ and $s \in [0,1]$, we just have to remember that $(x, 1) \equiv *$ for all x.

It is not difficult, then, to find a homotopy $H\colon CX \times I \to CX$ between id_{CX} and the constant maps c_*, that contracts the entire cone to a single point, the top of the cone $*$. All we need to do is define a homotopy between $\mathrm{id}_{X\times I}$ and the projection $p_1\colon X \times I \to X \times I$, $(x, s) \mapsto (x, 1)$, that maps the entire $X \times I$ onto the interval $X \times \{1\}$. If we make sure that the homotopy H remains fixed on $X \times \{1\}$ for all t, then the composition of H with the quotient map that identifies all points $(x, 1)$ with $*$ will be well-defined, and this will be the contraction of CX to $*$ that we are looking for. Let

$$H((x, s), t) = (1 - t)(x, s) + t(x, 1) = (1 - t)(x, s) + t \cdot *.$$

It is clear that

$$H((x, s), 0) = (x, s) = \mathrm{id}_{X\times I}(x, s),$$
$$H((x, s), 1) = * = c_*(x, s)$$

and

$$H((x, 1), t) = (1 - t)(x, 1) + t(x, 1) = (x, 1),$$

as required.

2. Geometry

Solution 19.

a. Using the cross product formula we can calculate the area of the triangle ΔABC as
$$S = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|.$$
The points A, B and C are collinear if and only if this area is equal to 0. To calculate this expression we need multiplication, addition and square roots. Of course we can omit the factor of $\frac{1}{2}$ and the square root and instead directly compare $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$ to 0. This method only works for $n = 3$ where the cross product can be defined, and $n = 2$ which can be considered as a special case with the third component of all vectors equal to 0.

b. Heron's formula states that the area of a triangle ABC with the semiperimeter
$$s = \frac{1}{2}(\|AB\| + \|BC\| + \|CA\|)$$
is
$$\sqrt{(s - \|AB\|)(s - \|BC\|)(s - \|CA\|)}.$$
Again, this area is 0 if and only if A, B and C are collinear. This method works for all n. We need multiplication, addition and square roots. We can skip the final

square root and instead compare $(s - \|AB\|)(s - \|BC\|)(s - \|CA\|)$ to 0 directly. Of course this product will be 0 if at least one of the factors is 0. For example,

$$s - \|AB\| = \frac{1}{2}(\|AB\| + \|BC\| + \|CA\|) - \|AB\| = \frac{1}{2}(\|BC\| + \|CA\| - \|AB\|),$$

and this is 0 if and only if $\|BC\| + \|CA\| - \|AB\| = 0$. Compare this to the solution using triangle inequalities.

c. For any vectors $\vec{a}$ and $\vec{b}$ we have

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\varphi.$$

We can use this formula to calculate the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\cos\varphi = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|}.$$

If the cosine is equal to 1 or -1, then the three points are collinear. This method works for all n. We need multiplication, addition and square roots. We can avoid the square roots if we calculate

$$\frac{(\overrightarrow{AB} \cdot \overrightarrow{AC})^2}{\|\overrightarrow{AB}\|^2\|\overrightarrow{AC}\|^2}$$

instead. The points are collinear if and only if this fraction is equal to 1.

d. The points A, B and C are collinear if and only if we can write $\overrightarrow{AC} = k \cdot \overrightarrow{AB}$ for some $k \in \mathbb{R}$. So, we can write

$$\overrightarrow{AB} = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad \overrightarrow{AC} = (y_1, y_2, \ldots, y_n)$$

and calculate

$$k_1 = \frac{y_1}{x_1}, \ \ldots, \ k_n = \frac{y_n}{x_n}.$$

The points are collinear if and only if $k_1 = \ldots = k_n$. This works for all n, but only if $\overrightarrow{AB}$ has no coordinates equal to 0, which makes this method impractical. We can improve on it if, instead of calculating the k_i directly, we solve an overdetermined system

$$\begin{aligned} x_1 k &= y_1, \\ &\vdots \\ x_n k &= y_n, \end{aligned}$$

of n equations with one unknown k. Methods for solving overdetermined systems of linear equations exist that give an approximate solution if an exact solution does not exist (for example, due to noise in the data). We can use the value of k to determine the order of the points on the line. See the next example for details.

e. A line through the points A and B has the direction vector $\overrightarrow{AB}$ and can be parametrized as

$$p\colon \vec{r}_A + t \cdot \overrightarrow{AB}, \quad t \in \mathbb{R}.$$

If $\vec{r}_A = (a_1, \ldots, a_n)$ and $\vec{r}_B = (b_1, \ldots, b_n)$, then

$$\overrightarrow{AB} = (b_1 - a_1, \ldots b_n - a_n)$$

and the points on the line p are of the form

$$(a_1, \ldots, a_n) + t(b_1 - a_1, \ldots b_n - a_n).$$

The point C, $\vec{r}_C = (c_1, \ldots, c_n)$, lies on the line through A and B if and only if the system

$$\begin{aligned} a_1 + t(b_1 - a_1) &= c_1, \\ a_2 + t(b_2 - a_2) &= c_2, \\ &\vdots \\ a_n + t(b_n - a_n) &= c_n, \end{aligned}$$

of n linear equations with one unknown t has a solution. Note that this is the same system as in the previous example. If $0 \le t \le 1$, then C lies between A and B. If $-\infty < t < 0$, the order of the points on the line is C, A, B. If $1 < t < \infty$, then the order is A, B, C.

f. In a triangle ΔABC we have

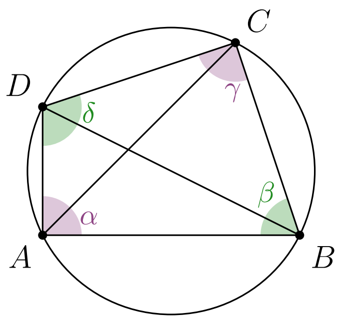
$$\|AB\| \le \|AC\| + \|CB\|, \quad \|BC\| \le \|BA\| + \|AC\| \quad \text{and} \quad \|AC\| \le \|AB\| + \|BC\|.$$

The points A, B and C are collinear if and only if one of the three inequalities is an equality. If $\|AB\| = \|AC\| + \|CB\|$, then the point C lies between A and B. If $\|BC\| = \|BA\| + \|AC\|$, then A lies between B and C. If $\|AC\| = \|AB\| + \|BC\|$, then B lies between A and C. This works for all n and the distances are easy to calculate. We need multiplication, addition and square roots.

Solution 20.

a. A quadrilateral $ABCD$ is cyclic if and only if the opposite angles are supplementary:

$$\alpha + \gamma = \beta + \delta = \pi.$$



Since the sum of the angles is equal to 2π, we only need to check if

$$\alpha + \gamma = \pi,$$

for example. To calculate the angle $\alpha = \sphericalangle BAD$, we can use the formula for the scalar product of vectors,

$$\overrightarrow{AB} \cdot \overrightarrow{AD} = \|\overrightarrow{AB}\| \|\overrightarrow{AD}\| \cos\alpha,$$

or we can use the cosine theorem

$$\|\overrightarrow{BD}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{AD}\|^2 - 2\|\overrightarrow{AB}\| \|\overrightarrow{AD}\| \cos\alpha.$$

We need multiplication, addition and square roots to calculate $\cos\alpha$ and $\cos\gamma$. We can use these to values to calculate

$$\begin{aligned} \cos(\alpha + \gamma) &= \cos\alpha\cos\gamma - \sin\alpha\sin\gamma = \\ &= \cos\alpha\cos\gamma - \sqrt{1 - (\cos\alpha)^2}\sqrt{1 - (\cos\gamma)^2}. \end{aligned}$$

The sum of the opposite angles will be equal to π if and only if

$$\cos\alpha\cos\gamma - \sqrt{1-(\cos\alpha)^2}\sqrt{1-(\cos\gamma)^2} = -1$$

and we have avoided the use of arccos. This works if the quadrilateral is convex, that is if $\alpha, \beta, \gamma, \delta \in (0, \pi)$. If $\alpha + \gamma > \pi$, then the point C will lie inside the circumcircle of the triangle ABC. If $\alpha + \gamma < \pi$, then the point C will lie outside the circumcircle of the triangle ABC. We cannot tell which is the case only from $\cos(\alpha + \gamma)$, but if the quadrilateral $ABCD$ is convex, we can calculate

$$\begin{aligned}\sin(\alpha+\gamma) &= \sin\alpha\cos\gamma + \cos\alpha\sin\gamma = \\ &= \sqrt{1-(\cos\alpha)^2}\cos\gamma + \cos\alpha\sqrt{1-(\cos\gamma)^2}\end{aligned}$$

If $\sin(\alpha + \gamma) > 0$, then $\alpha + \gamma < \pi$. If $\sin(\alpha + \gamma) < 0$, then $\alpha + \gamma > \pi$.

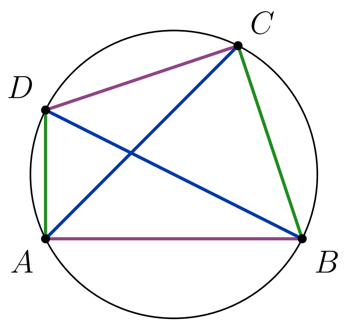
b. Let $A = (x_A, y_A)$, $B = (x_B, y_B)$ and $C = (x_C, y_C)$ be three vertices of the quadrilateral $ABCD$ ordered in the positive direction (counterclockwise). Then the point $D = (x_D, y_D)$ lies on the circumcircle of the triangle ABC if and only if

$$\Delta = \begin{Vmatrix} x_A & y_A & x_A^2 + y_A^2 & 1 \\ x_B & y_B & x_B^2 + y_B^2 & 1 \\ x_C & y_C & x_C^2 + y_C^2 & 1 \\ x_D & y_D & x_D^2 + y_D^2 & 1 \end{Vmatrix} = 0.$$

If $\Delta > 0$, then D lies inside the circumcircle. If $\Delta < 0$, then D lies outside the circumcircle.

c. Ptolemy's theorem states that a quadrilateral $ABCD$ is cyclic if and only if

$$\|AB\|\|CD\| + \|BC\|\|AD\| = \|AC\|\|BD\|.$$



Unfortunately, Ptolemy's inequality states that

$$\|AB\|\|CD\| + \|BC\|\|AD\| \geq \|AC\|\|BD\|$$

for all quadrilaterals $ABCD$, so we cannot use the sign of

$$\|AB\|\|CD\| + \|BC\|\|AD\| - \|AC\|\|BD\|$$

to decide whether the fourth point lies inside the circumcircle through the other three. This is also obvious due to symmetry, since the points D and B lie inside the circumcircles of the triangles ABC and ACD, respectively, if and only if the points C and A do not lie inside the circumcircles of the triangles ABD and BCD, respectively. Ptolemy's theorem is symmetrical with respect to the diagonals and cannot detect which of these two possibilities occurs for a non-cyclic quadrilateral.

d. If we know the centre S and the radius r of the circumcircle $\mathcal{C}$ of the triangle ABC, the value of $h(D, \mathcal{C})$ will tell us if D lies on $\mathcal{C}$, inside of $\mathcal{C}$ or outside of $\mathcal{C}$. The problem of finding S and r is addressed in the next exercise.

Is there something we can say about the position of the point D without calculating $\mathcal{C}$? Let us state a slightly modified version of the Power-of-a-Point Theorem (with signs). Let the line through P intersect the circle $\mathcal{C}$ at A and B. Then

$$h(P,\mathcal{C}) = \|PA\|\|PB\| \cdot \operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PB}),$$

where

- $\operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PB}) = 1$ if $\overrightarrow{PA}$ and $\overrightarrow{PB}$ point in the same direction,
- $\operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PB}) = -1$ if $\overrightarrow{PA}$ and $\overrightarrow{PB}$ point in opposite directions and
- $\operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PB}) = 0$ if $P = A$ or $P = B$.

(Power-of-a-Point Theorem) Let $ABCD$ be a quadrilateral and let P denote the intersection of the diagonals AC and BD. Then A, B, C and D are cocyclic if and only if

$$\|PA\|\|PC\| \operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PC}) = \|PB\|\|PD\| \operatorname{sgn}(\overrightarrow{PB}, \overrightarrow{PD}).$$

Now, imagine that $\|PA\|\|PC\| \operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PC}) \neq \|PB\|\|PD\| \operatorname{sgn}(\overrightarrow{PB}, \overrightarrow{PD})$. We then know that D does not lie on the circumcircle of the triangle ABC. There exist exactly two points on the line through P, B and D, we will denote them by D' and D'', such that

$$\|PA\|\|PC\| = \|PB\|\|PD'\| \quad \text{and} \quad \|PA\|\|PC\| = \|PB\|\|PD''\|.$$

Indeed, these two points are the intersections of the circle centered at P with radius

$$r = \frac{\|PA\|\|PC\|}{\|PB\|}$$

with the line through P, B and D. Exactly one of these points, call it D' is such that

$$\|PA\|\|PC\| \operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PC}) = \|PB\|\|PD'\| \operatorname{sgn}(\overrightarrow{PB}, \overrightarrow{PD'}).$$

By the Power-of-a-Point Theorem the point D' is the intersection of the line through P, B and D and the circumcircle of the triangle ABC, but we do not need to calculate the equation of the circumcircle to compute D'! We can get its coordinates by computing

$$\vec{r}_{D'} = \vec{r}_P \pm \frac{r}{\|PB\|} \cdot \overrightarrow{PB} = \vec{r}_P \pm \frac{\|PA\|\|PC\|}{\|PB\|^2} \cdot \overrightarrow{PB},$$

choosing the sign so that $\operatorname{sgn}(\overrightarrow{PB}, \overrightarrow{PD'}) = \operatorname{sgn}(\overrightarrow{PA}, \overrightarrow{PC})$.

It is now easy to see that the point D lies inside the circumcircle of the triangle ABC if and only if it lies on the segment BD'. We need to solve the system $\overrightarrow{BD} = k\overrightarrow{BD'}$ of n equations with one unknown k. If $0 < k < 1$, then D lies inside the circumcircle of triangle ABC.

e. See the next problem for several different ways of computing the equation of the circumcircle.

f. Brahmagupta's formula states that the area of a cyclic quadrilateral with the semiperimeter

$$s = \frac{1}{2}(\|AB\| + \|BC\| + \|CD\| + \|AD\|)$$

is

$$S = \sqrt{(s - \|AB\|)(s - \|BC\|)(s - \|CD\|)(s - \|AD\|)}.$$

We can use the determinant formula to calculate the areas S_{ABC} and S_{ACD} of triangles ABC and ACD. If $S_{ABC} + S_{ACD} \neq S$, then $ABCD$ is not a cyclic quadrilateral. We could also calculate the area of the quadrilateral directly using Coolidge's formula

$$S=\sqrt{(s-\|AB\|)(s-\|BC\|)(s-\|CD\|)(s-\|AD\|)-\frac{1}{4}(ac+bd+pq)(ac+bd-pq)}$$

and compare this to Brahmagupta's formula to see if the quadrilateral is cyclic. Unfortunately, neither of these two comparisons can tell us if the point D lies inside or outside of the circumcircle $ABCD$, because the expression for S is symmetric with respect to the the four points while the problem of whether the fourth vertex lies inside the circle through the other three is obviously not symmetric. In fact, Bretschneider's formula gives us the area of any quadrilateral with sides a, b, c and d as

$$\begin{aligned} S &= \sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\alpha+\gamma} = \\ &= \sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\beta+\delta}. \end{aligned}$$

We see that cyclic quadrilateral is where this expression reaches its maximum. The area will decrease as soon as the fourth point does not lie on the circle, regardless of whether it lies inside or outside.

Solution 21.

a. Let $A=(x_A,y_A)$, $B=(x_B,y_B)$, $C=(x_C,y_C)$ and $S=(x_S,y_S)$. Then the condition $d(A,S)=d(B,S)=d(C,S)$ gives us a system

$$\sqrt{(x_S-x_A)^2+(y_S-y_A)^2}=\sqrt{(x_S-x_B)^2+(y_S-y_B)^2}=\sqrt{(x_S-x_B)^2+(y_S-y_B)^2}$$

of two equations with two unknowns x_S and y_S. We can omit the square roots and write

$$\begin{aligned} (x_S-x_A)^2+(y_S-y_A)^2 &= (x_S-x_B)^2+(y_S-y_B)^2, \\ (x_S-x_A)^2+(y_S-y_A)^2 &= (x_S-x_C)^2+(y_S-y_C)^2. \end{aligned}$$

Squaring all the terms and cancelling duplicates we get

$$\begin{aligned} -2x_Sx_A+{x_A}^2-2y_Sy_A+{y_A}^2 &= -2x_Sx_B+{x_B}^2-2y_Sy_B+{y_B}^2, \\ -2x_Sx_A+{x_A}^2-2y_Sy_A+{y_A}^2 &= -2x_Sx_C+{x_C}^2-2y_Sy_C+{y_C}^2. \end{aligned}$$

After a bit of rearranging we get

$$\begin{aligned} x_S(x_B-x_A)-y_S(y_A-y_B) &= \frac{1}{2}({x_B}^2+{y_B}^2-{x_A}^2-{y_A}^2), \\ x_S(x_C-x_A)-y_S(y_A-y_C) &= \frac{1}{2}({x_C}^2+{y_C}^2-{x_A}^2-{y_A}^2). \end{aligned}$$

This is a linear system, so it not too difficult to compute a solution. For $n>2$ we would still only get 2 equations but n unknowns and the solution to the system would be a $(n-2)$-dimensional affine subspace of points, equidistant from A, B and C.

b. Let $\vec{r}_A=A=(x_A,y_A)$, $\vec{r}_B=B=(x_B,y_B)$, $\vec{r}_C=C=(x_C,y_C)$ and let L, M and N be the bisectors of the segments BC, AC and AB, respectively. Then

$$\vec{r}_L=\frac{1}{2}(\vec{r}_A+\vec{r}_B),\quad \vec{r}_M=\frac{1}{2}(\vec{r}_B+\vec{r}_C)\quad\text{and}\quad \vec{r}_N=\frac{1}{2}(\vec{r}_A+\vec{r}_C).$$

These will be the points on the bisectors l, m and n. Now, we have to find the vectors $\vec{l}$, $\vec{m}$ and $\vec{n}$, perpendicular to the vectors $\vec{a}=\overrightarrow{BC}$, $\vec{b}=\overrightarrow{AC}$ and $\vec{c}=\overrightarrow{AB}$, in this order. If $n=2$ we have

$$\begin{aligned}\vec{a} &= \vec{r}_C - \vec{r}_B &&= (x_C - x_B, y_C - y_B),\\ \vec{b} &= \vec{r}_C - \vec{r}_A &&= (x_C - x_A, y_C - y_A),\\ \vec{c} &= \vec{r}_B - \vec{r}_A &&= (x_B - x_A, y_B - y_A),\end{aligned}$$

so we can choose

$$\begin{aligned}\vec{l} &= (y_C - y_B, -(x_C - x_B)),\\ \vec{m} &= (y_C - y_A, -(x_C - x_A)),\\ \vec{n} &= (y_B - y_A, -(x_B - x_A)).\end{aligned}$$

For general n we can use the fact that the circumcentre must lie in the plane determined by the points A, B and C, so we can write

$$\begin{aligned}\vec{l} &= \alpha_l \vec{a} + \beta_l \vec{b},\\ \vec{m} &= \alpha_m \vec{a} + \beta_m \vec{b},\\ \vec{n} &= \alpha_n \vec{a} + \beta_n \vec{b},\end{aligned}$$

for some $\alpha_l, \beta_l, \alpha_m, \beta_m, \alpha_n, \beta_n$. The conditions $\vec{l}\cdot\vec{a} = \vec{m}\cdot\vec{b} = \vec{n}\cdot\vec{c} = 0$ mean we have a system of 3 equations with 6 unknowns, which makes sense because each directional vector of the bisector is only determined up to a scalar factor. Once we solve this system in general and choose the three parameters to obtain one particular solution $\vec{l}$, $\vec{m}$, $\vec{n}$, all is left is to find the intersection of the lines

$$l\colon \vec{r}_L + s\cdot\vec{l}, \quad m\colon \vec{r}_M + t\cdot\vec{m}, \quad \text{and} \quad n\colon \vec{r}_N + u\cdot\vec{n}.$$

We get a system of equations

$$\begin{aligned}\vec{r}_L + s\cdot\vec{l} &= \vec{r}_M + t\cdot\vec{m},\\ \vec{r}_L + s\cdot\vec{l} &= \vec{r}_N + u\cdot\vec{n}.\end{aligned}$$

This is an overdetermined system of $2n$ equations, two for each coordinate, with 3 unknowns, the parameters s, t and u.

c. Using the same notation as in the previous case, we can find the coordinates of the point $L = \left(\frac{1}{2}(x_C - x_B), \frac{1}{2}(y_C - y_B)\right)$. From the directional vector $\vec{l} = (y_C - y_B, -(x_C - x_B))$ we can compute the slope of the line l,

$$k_l = -\frac{x_C - x_B}{y_C - y_B}.$$

The equation of the line l in Cartesian coordinates is $y - y_L = k_l(x - x_L)$ or

$$y = -\frac{x_C - x_B}{y_C - y_B}x + \frac{x_C - x_B}{y_C - y_B}\cdot\frac{x_B + x_C}{2} + \frac{y_B + y_C}{2}.$$

We get similar equations for the other two lines and together they form, in principle, a system of three equations with 2 unknowns, x and y. However, if we multiply the equation given above by $y_C - y_B$, we get

$$(y_C - y_B)y = -(x_C - x_B)\,x + \frac{1}{2}(x_C - x_B)(x_B + x_C) + \frac{1}{2}(y_B + y_C)(y_C - y_B)$$

or, after a quick rearrangement

$$(x_C - x_B)x + (y_C - y_B)y = \frac{1}{2}(x_C^2 + y_C^2 - x_B^2 - y_B^2).$$

The other two equations are

$$(x_C - x_A)x + (y_C - y_A)y = \frac{1}{2}(x_C^2 + y_C^2 - x_A^2 - y_A^2)$$

and

$$(x_B - x_A)x + (y_B - y_A)y = \frac{1}{2}(x_B^2 + y_B^2 - x_A^2 - y_A^2).$$

We see that second and the third equation are the same as in the first solution and the first is their difference so it does not add anything to the system. We have, in fact, obtained the exact same system of two equations with two unknowns as in the first solution.

Solution 22.

a. The volume of the tetrahedron spanned by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ is

$$V = \frac{1}{6}\|\vec{a} \cdot (\vec{b} \times \vec{c})\|.$$

This volume will be equal to 0 if and only if the points A, B, C and D are coplanar. The formula uses the cross product so it only works for $n = 3$. If we use the Heron-type formula instead, we only need the lengths of the six edges, so it works for all $n \geq 3$.

b. We need to consider the overdetermined system

$$\overrightarrow{AD} = k_1\overrightarrow{AB} + k_2\overrightarrow{AC}$$

of $n \geq 3$ linear equations with two unknowns, k_1 and k_2. If a solution exists, then the four points lie on the same plane.

c. If $n = 3$, then the plane containing points A, B, C will have the normal vector

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

and the equation

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{r}_A.$$

So, we need to check if

$$\vec{n} \cdot \vec{r}_D = \vec{n} \cdot \vec{r}_A.$$

This does not work for $n \neq 3$.

d. If $\overrightarrow{AB} = k \cdot \overrightarrow{CD}$, then the lines AB and CD are parallel and the four points line on the same plane. To check this, we need to solve an overdetermined system of n equations with one unknown k, so it would be easier to use the second solution given above. If we wish to proceed and this system does not have a solution, then the two lines are not parallel and might intersect. We can parametrize the lines AB and CD as

$$\vec{r}_A + k_1 \cdot \overrightarrow{AB} \quad \text{and} \quad \vec{r}_C + k_2 \cdot \overrightarrow{CD}.$$

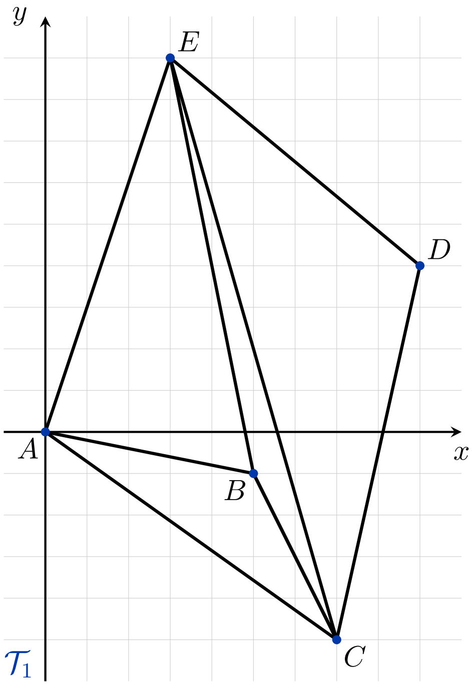
A point in the intersection would satisfy all of these equations, which gives us an overdetermined system of $2n$ equations with 2 unknowns, k_1 and k_2. If a solution exists, then the four points lie in the same plane. If this system does not have a solution, then these points are not coplanar.

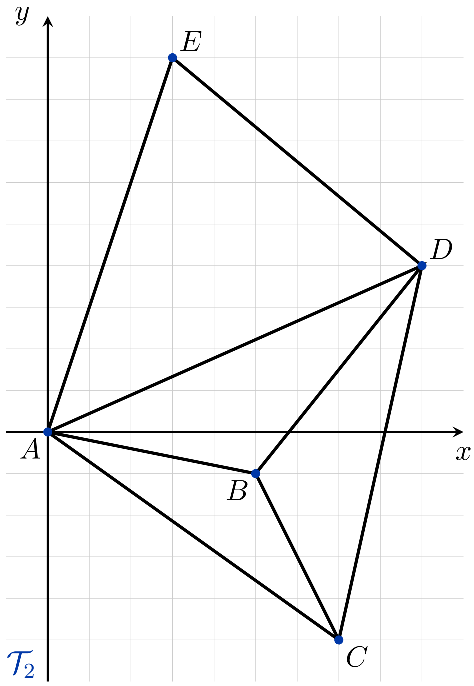
3. Triangulations and simplicial complexes

SOLUTION 23.

a. A vertical line sweeping to the right meets the points A, E, B, C and D in order. We connect E to A, B to A and E, C to A, E and B and finally D to E and C.

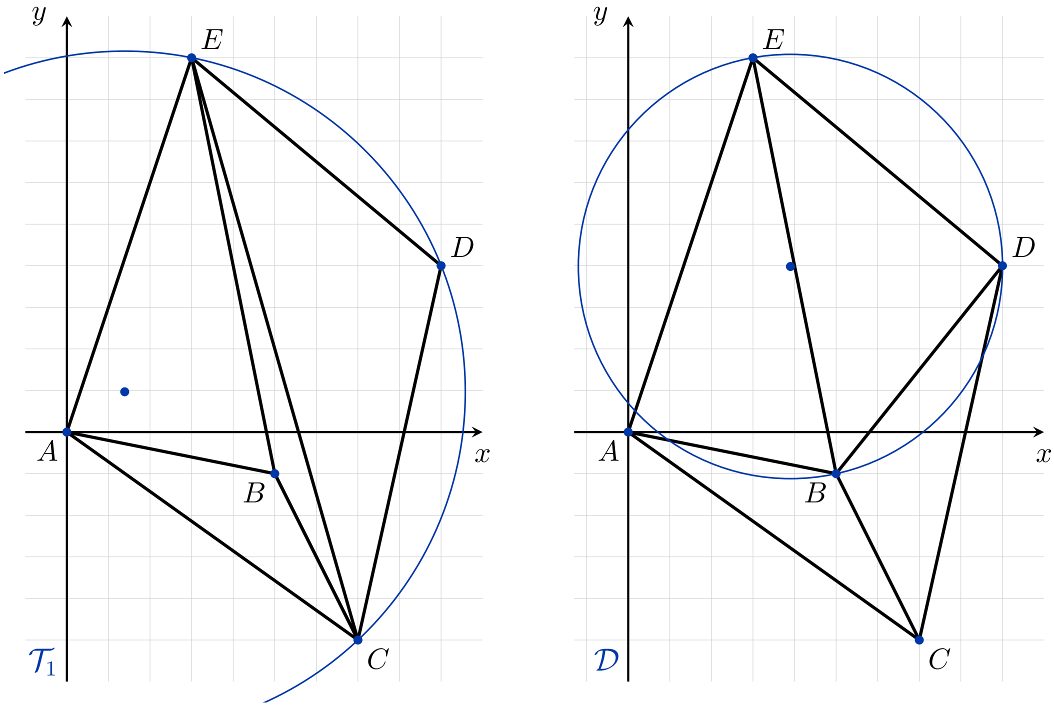
A horizontal line moving upwards meets the points C, B, A, D and E in this order. We first connect B to C, then A to C and B, D to C, B and A and finally E to A and D.



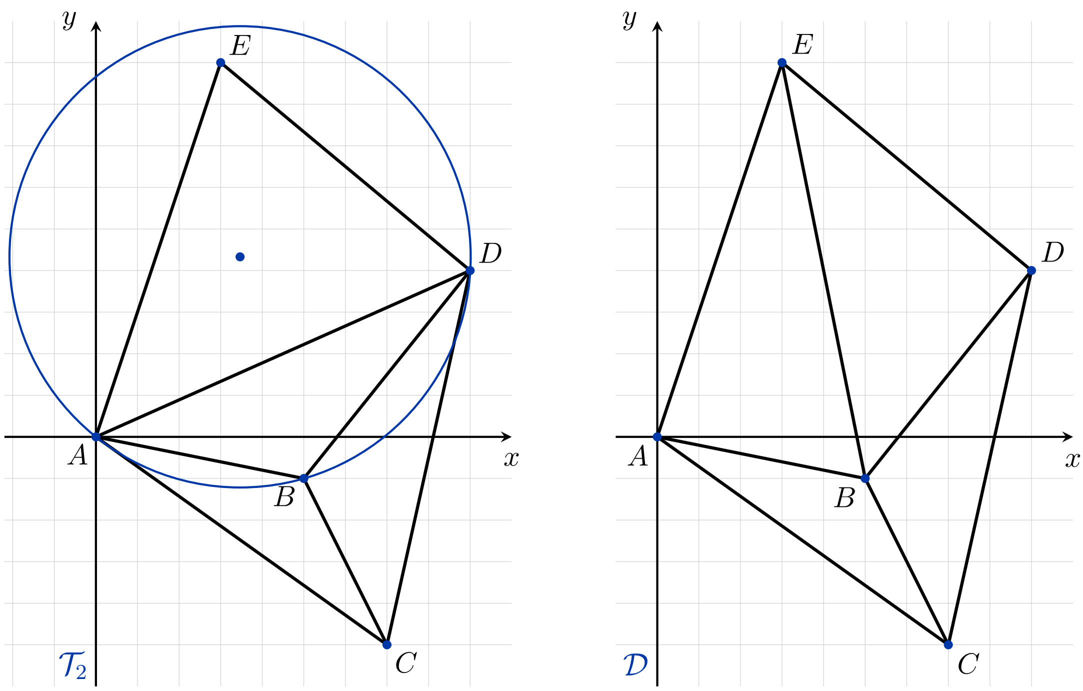


The triangles in $\mathcal{T}_1$ are ABC, ABE, BCE and CDE. The triangles in $\mathcal{T}_2$ are ABC, ABD, ADE and BCD.

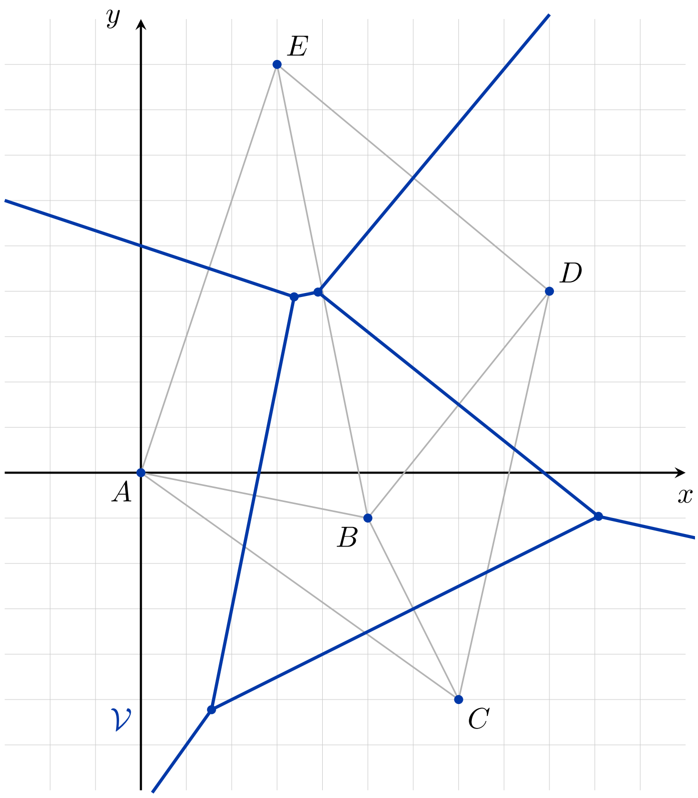
b. In order to determine what edges to flip, we draw some of the circumcircles of the triangles in $\mathcal{T}_1$. For example, the edge CE looks suspicious, so we draw the circumcircle of the triangle CDE. We see that the point B lies inside this circumcircle, so we should replace the edge CE by the edge BD, replacing the triangles BCE and CDE by the triangles BCD and BDE. When we draw the circumcircle of the triangle BDE, we see that the point C lies outside, so BD is a Delaunay edge. The circumcircle of the triangle BDE also does not contain the point A, which means that we should not replace the edge BE by the edge AD since BE is already a Delaunay edge. In fact, the triangles in the Delaunay triangulation $\mathcal{D}$ of S are ABC, ABE, BCD and BDE.



In the case of $\mathcal{T}_2$ consider the quadrilateral $ABDE$. Should we exchange the edge AD for the edge BE? The circumcircle of the triangle ABD contains E, so the answer is yes. Once the flip is made, the new triangles are ABC, ABE, BCD and BDE and we have once more obtained the Delaunay triangulation of S after a single flip.

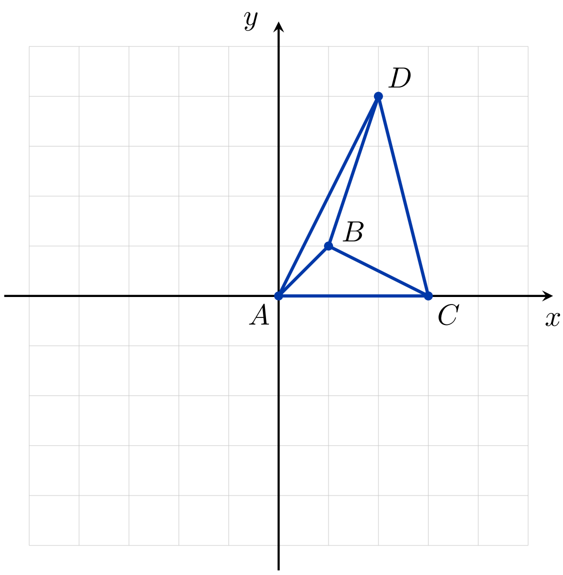


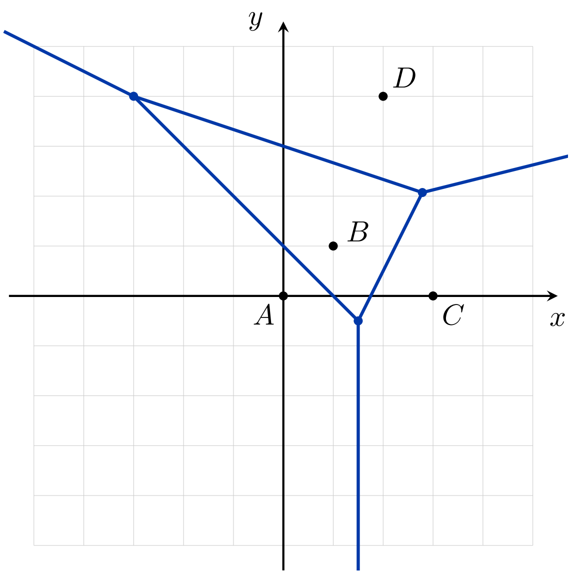
c. To construct the Voronoi diagram corresponding to the Delaunay triangulation $\mathcal{D}$, we need to find the circumcentres of the triangles ABC, ABE, BCD and BDE. We connect two circumcentres if and only if the corresponding two triangles share an edge. The infinite edges run from the corresponding circumcentres along the segment bisectors of the boundary edges.



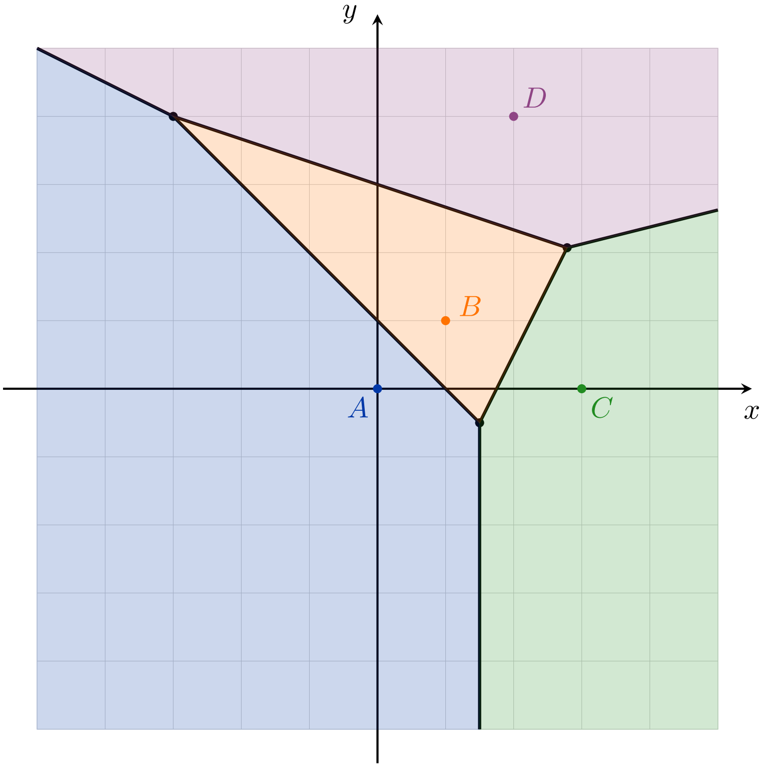
SOLUTION 24.

We are looking for the Voronoi diagram of the set $S = \{A, B, C, D\}$. We begin by finding a triangulation of S using horizontal line sweep. Since the points A and C lie on the same horizontal line, we need to decide which goes first. Let us suppose the points should be ordered left-to-right in such cases. Then the order is A, C, B, D. We add the edges AC, AB, BC, AD, CD and BD in this order. The triangles in the triangulation are ABC, ABD and BCD. There are no convex quadrilaterals so no diagonals can be flipped and this is already the Delaunay triangulation.

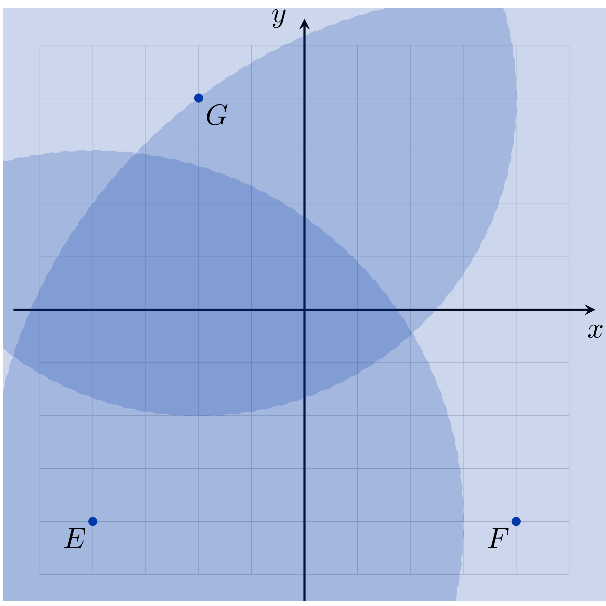




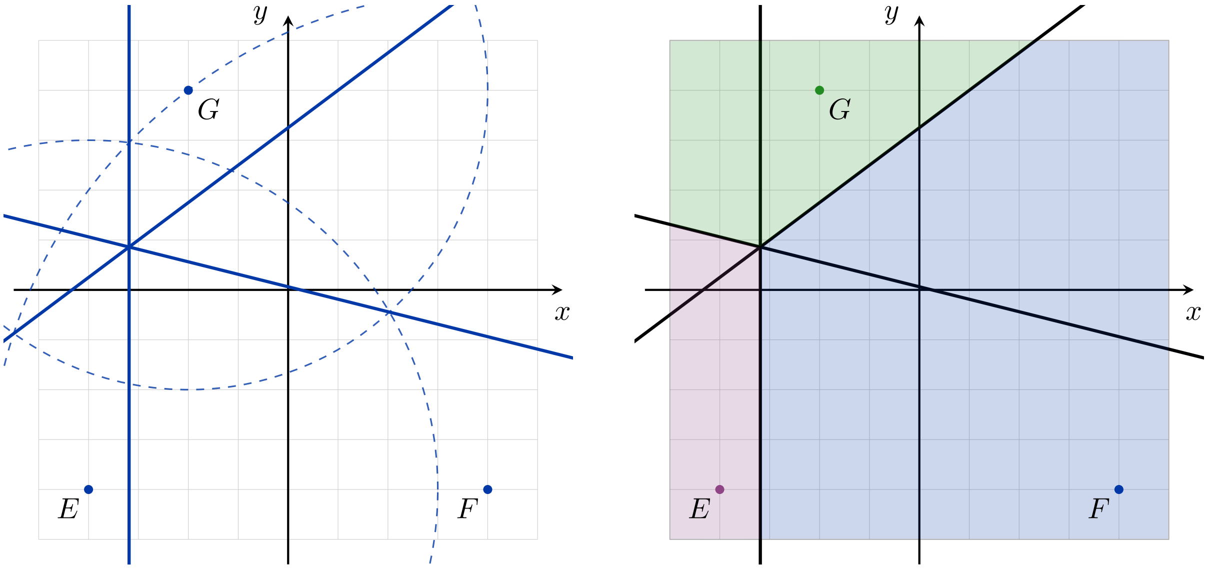
Next, we need to find the circumcentres of the three triangles and connect them. The infinite Voronoi edges start at the corresponding circumcentre and follow along the bisectors of the outside edges of the triangulation. The Voronoi edges divide the $[-5,5] \times [-5,5]$ square into four regions, each served by the closest point.



If we draw the circles with the corresponding radii, we see that the three Mercury post centers can indeed cover the entire $[-5,5] \times [-5,5]$ square.



To divide the area into polygonal cells, we will use the power lines of these circles. The power line of each pair of circles passes through their non-empty intersection and all three power lines intersect in a single point, the power center of the three circles. The corresponding areas are shown in the figure on the right.



SOLUTION 25.

For the open stars we need to list all simplices that have the given simplex as a face, so

$$\begin{array}{rcl} \mathrm{st}(A) & = & \{A, AB, AD, AE, ABD\}, \\ \mathrm{st}(AB) & = & \{AB, ABD\}. \end{array}$$

It is also easy to calculate

$$\begin{array}{rcl} \mathrm{cl}(\mathrm{st}(A)) & = & \{A, B, D, E, AB, AD, AE, BD, ABD\}, \\ \mathrm{cl}(\mathrm{st}(AB)) & = & \{A, B, D, AB, AD, BD, ABD\}. \end{array}$$

Since $\mathrm{cl}(A) = \{A\}$, we get

$$\mathrm{st}(\mathrm{cl}(A)) = \mathrm{st}(A) = \{A, AB, AD, AE, ABD\}.$$

On the other hand, $\mathrm{cl}(AB) = \{A, B, AB\}$, so

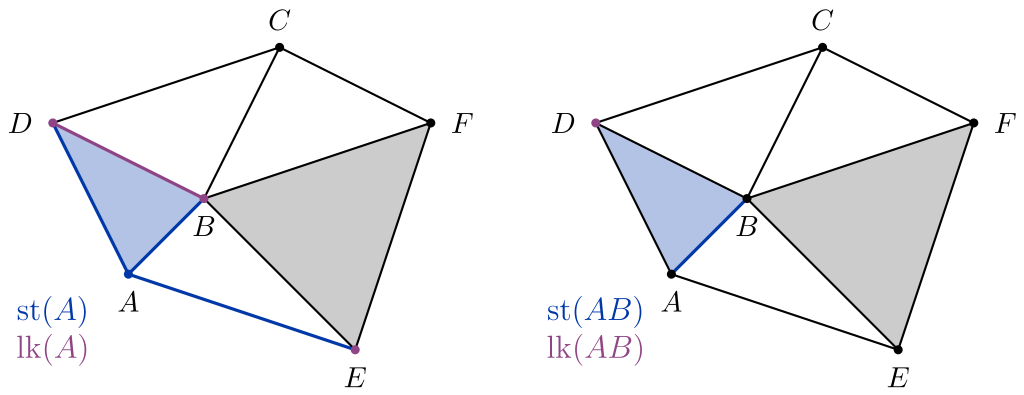
$$\begin{array}{rcl} \mathrm{st}(\mathrm{cl}(AB)) & = & \mathrm{st}(\{A, B, AB\}) = \\ & = & \mathrm{st}(A) \cup \mathrm{st}(B) \cup \mathrm{st}(AB). \end{array}$$

We have $\mathrm{st}(B) = \{B, AB, BC, BD, BE, BF, ABD, BEF\}$, so

$$\mathrm{st}(\mathrm{cl}(AB)) = \{A, B, AB, AD, AE, BC, BD, BE, BF, ABD, BEF\}.$$

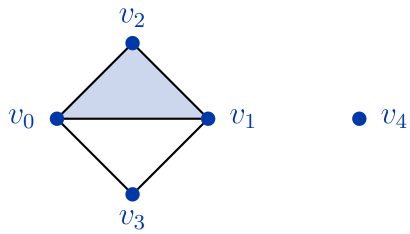
From here, we get

$$\begin{array}{rcl} \mathrm{lk}(A) & = & \mathrm{cl}(\mathrm{st}(A)) \setminus \mathrm{st}(\mathrm{cl}(A)) = \\ & = & \{B, D, E, BD\}, \\ \mathrm{lk}(AB) & = & \mathrm{cl}(\mathrm{st}(AB)) \setminus \mathrm{st}(\mathrm{cl}(AB)) = \\ & = & \{D\}. \end{array}$$

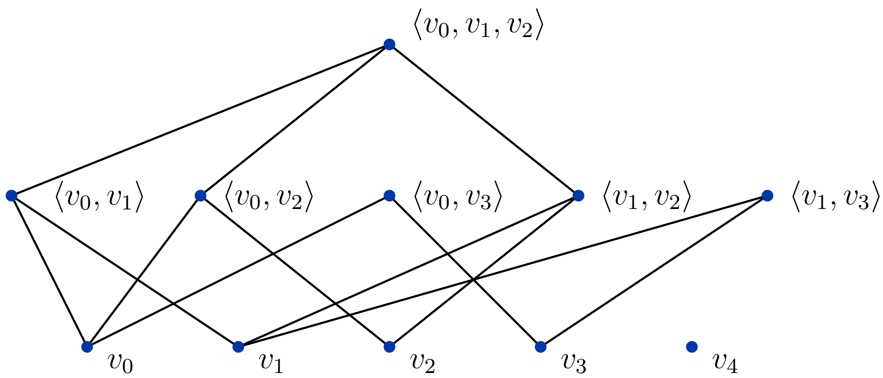


Solution 26.

a. We need to add the faces $\langle v_0, v_2 \rangle$ and $\langle v_1, v_2 \rangle$ of the 2-simplex $\langle v_0, v_1, v_2 \rangle$.



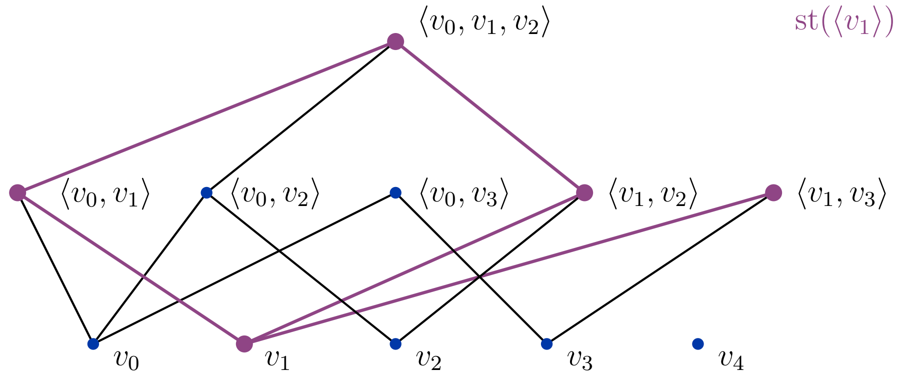
b. To construct the Hasse diagram, we divide the simplices into groups according to their dimensions and connect each k-simplex to all its $(k-1)$-dimensional faces.



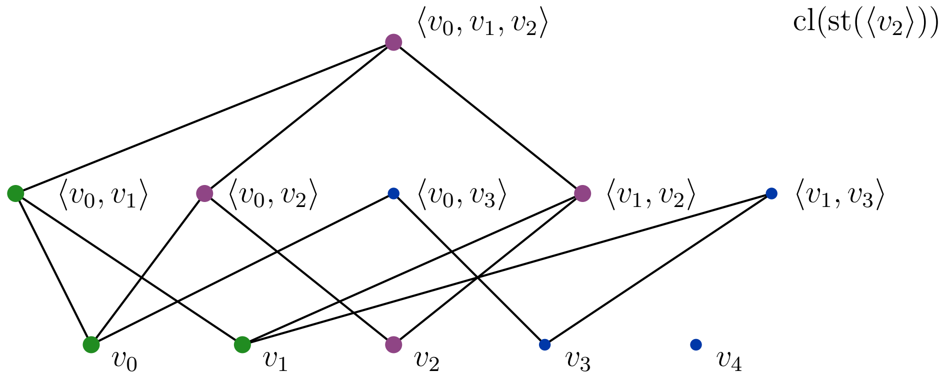
c. It is easy to see from the definitions that

$$
\begin{aligned}
\mathrm{st}(\langle v_1 \rangle) &= \{\langle v_1 \rangle, \langle v_0, v_1 \rangle, \langle v_1, v_2 \rangle, \langle v_1, v_3 \rangle, \langle v_0, v_1, v_2 \rangle\}, \\
\mathrm{st}(\langle v_1, v_3 \rangle) &= \{\langle v_1, v_3 \rangle\}, \\
\mathrm{lk}(\langle v_2 \rangle) &= \{\langle v_0 \rangle, \langle v_1 \rangle, \langle v_0, v_1 \rangle\}, \\
\mathrm{lk}(\langle v_0, v_3 \rangle) &= \{\}.
\end{aligned}
$$

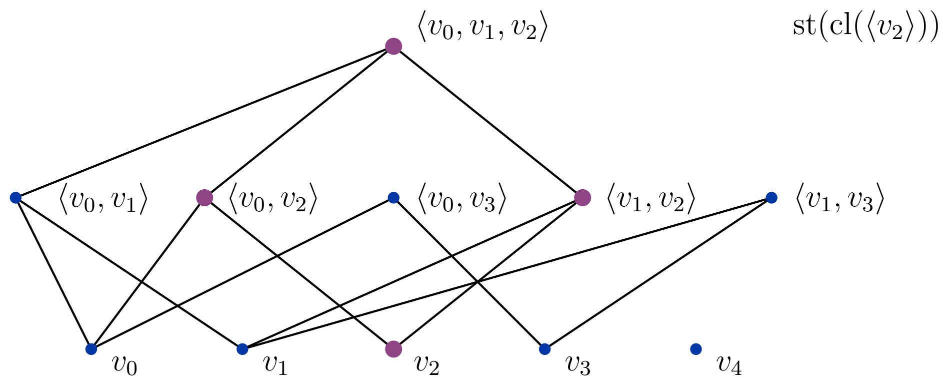
We can also find $\mathrm{st}(\sigma)$ by listing all the simplices that we can reach from σ by non-descending paths in the Hasse diagram. In the following figure the simplices in $\mathrm{st}(\langle v_1 \rangle)$ and the corresponding ascending paths are marked.



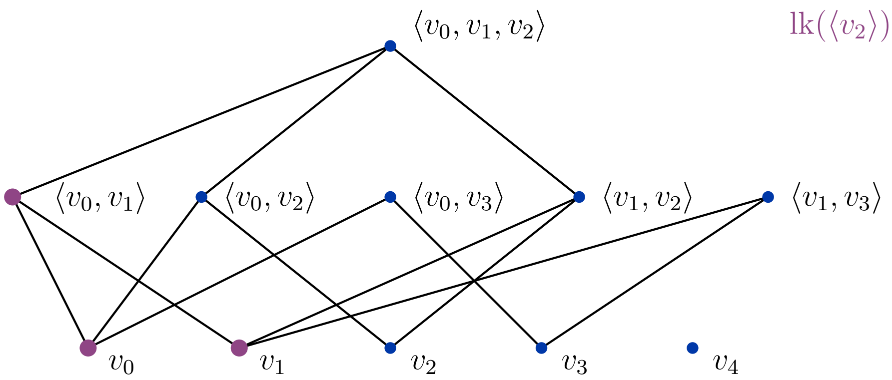
For $\mathrm{lk}(\sigma)$ we need to find $\mathrm{cl}(\mathrm{st}(\sigma))$, which means we need to include all possible destinations of desending paths starting from simplices we can reach by ascending from σ. In the following figure the simplices added during ascent are shown in purple and the simplices added during descent are shown in green. The union of both makes up $\mathrm{cl}(\mathrm{st}(\sigma))$.



We then need to exlude all simplices from $\mathrm{st}(\mathrm{cl}(\sigma))$, which we obtain by finding all possible destinations of ascending paths starting from simplices we can reach by descending from σ. In the following figure the simplices we add during descent are shown in purple (there is only one since the vertex $\langle v_2\rangle$ is already closed) and the simplices added during ascent are show in green. The union of both makes up $\mathrm{st}(\mathrm{cl}(\sigma))$.



Marking all points that are marked in the first picture but not the second we obtain $\mathrm{lk}(\sigma)$.



Solution 27.

a. We need to find the number of vertices c_0, edges c_1 and triangles c_2 for each of the given simplicial complexes. Then their Euler characteristic will be equal to

$$\chi = c_0 - c_1 + c_2.$$

For example, for A we obviously have $c_2(A) = 4$. The vertices of the given triangles are $1, 2, 3$ and 4, so $c_0(A) = 4$. Finally, the edges are $12, 13, 23, 14, 24, 34$, so $c_1(A) = 6$. We get

$$\chi(A) = 4 - 6 + 4 = 2.$$

Similarly, we get $c_2(B) = 5$, $c_0(B) = 7$ and the edges are

$$12, 13, 23, 14, 24, 25, 35, 26, 36, 37, 57,$$

so $c_1(B) = 11$. Therefore,

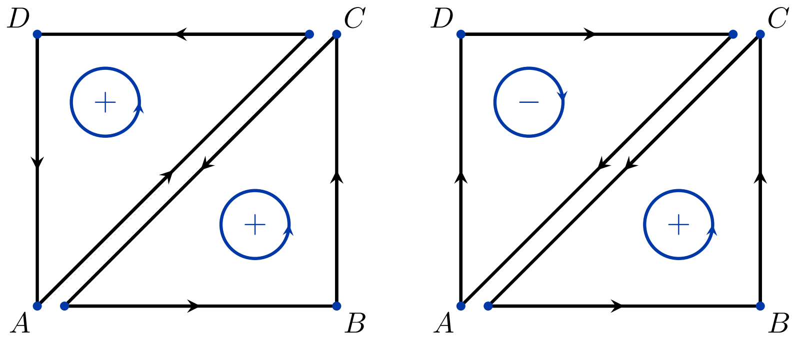
$$\chi(B) = 7 - 11 + 5 = 1.$$

The rest of the results are given in the table at the end.

b. To determine if a given 2-dimensional simplicial complex is a surface, we first need to count how many triangles each edge appears in. In a manifold without boundary, each edge will appear in exactly 2 triangles. In a manifold with boundary, the boundary edges will appear in exactly one of the triangles and the interior edges will appear in exactly two of the triangles. If an edge appears in more than two triangles, then that simplicial complex is not a manifold. For example, each of the six edges $12, 13, 23, 14, 24, 34$ of A appears in exactly 2 triangles. On the other hand, the edge 23 appears in three triangles of B, namely 123, 235 and 236. So, B is not a manifold. The rest of the results are given in the table at the end.

c. In the case of a surface each edge appears at most twice and a graph induced by exactly those edges that appear once will form the 1-dimensional boundary of the surface. We can use a simple program that counts the number of connected components in a graph to determine the number of boundary components. For example, the edges of C that appear exactly once are $13, 35, 15, 24, 46, 26$. It is easy to see that these edges belong to a graph, that has two components, namely the cycles $(1, 3, 5)$ and $(2, 4, 6)$. So, C has two boundary components. The rest of the results are summarized in the table at the end.

d. Once again it is not too difficult to write a short program that goes through the list of all triangles and tries to orient them consistently. If it can be done, then the surface is orientable. If not, then it is not orientable. Two triangles that share an edge, say ABC and ACD, are oriented consistently if the common edge appears with two different orientations. For example, the orientations (A, B, C) and (A, C, D) suggest the edges are AB, BC, CA and AC, CD, DA. Since AC and CA have different orientations, these two triangles are oriented consistently. If, on the other hand, the second triangle is oriented as (A, D, C), then the corresponding oriented edges are AD, DC and CA, so (A, B, C) and (A, D, C) are inconsistent orientations.



e. Since A is orientable with $\chi(A) = 2$ and $b = 0$, we have

$$2 = 2 - 2g - 0,$$

so the genus of A is 0. For C we have $\chi(C) = 0$ and $b = 2$, so

$$0 = 2 - 2g - 2$$

and the genus of C is also 0. The surface D is orientable, $\chi(D) = 0$ and $b = 0$, so

$$0 = 2 - 2g - 0$$

and the genus of D is 1. The rest of the surfaces are non-orientable. For E we have $\chi(E) = 0$ and it has no boundary, so

$$0 = 2 - g,$$

which implies that the genus of E is 2. On the other hand, the non-orientable G has a boundary component, so we cannot use this formula to define its genus. Finally, we have $\chi(H) = 1$ and it has no boundary components, so

$$1 = 2 - g$$

and the genus of H is 1.

f. We see that A is an orientable manifold of genus 0, so it can only be a sphere S^2. An orientable manifold C of genus 0 with two boundary components is a sphere with two discs cut out, which is to say a cylinder $S^1 \times I$. The third orientable manifold, D, has genus 1 and no boundary components, so it is a torus $S^1 \times S^1$. In the non-orientable case, E has genus 2 and is therefore a Klein bottle and H with genus 1 is the real projective plane $\mathbb{R}P^2$. With a bit of persistence, we can draw a picture of G which shows that G is a triangulated rectangle with two opposite edges glued together with a half-twist. So, G is a triangulated Moebius band.

genus	0	1	2
orientable	S^2	T	$T\#T$
non-orientable		P	$P\#P$

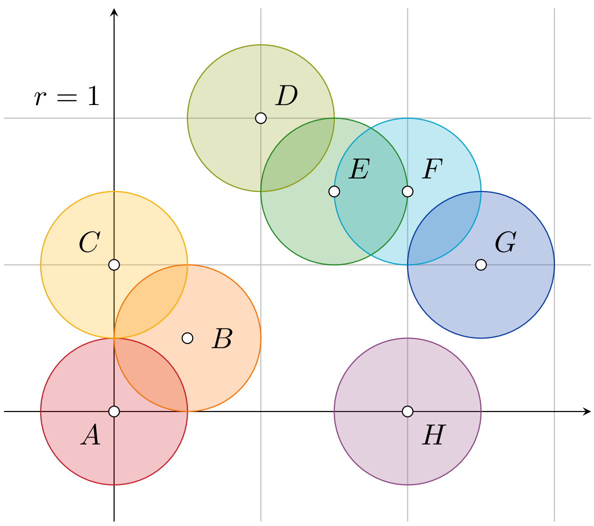
To summarize, we have:

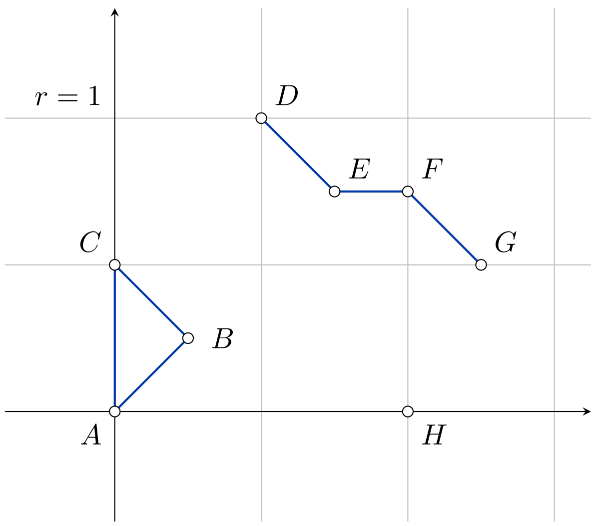
	Euler characteristic	manifold Y/N	# of boundary components	orientable Y/N	genus	name
A	2	Y	0	Y	0	S^2
B	1	N	-	-	-	-
C	0	Y	2	Y	0	$S^1 \times I$
D	0	Y	0	Y	1	$S^1 \times S^1$
E	0	Y	0	N	2	Klein bottle
F	1	N	-	-	-	-
G	0	Y	1	N	-	Moebius band
H	1	Y	0	N	1	$\mathbb{R}P^2$

4. Vietoris-Rips and Čech complexes

Solution 28.

In the Vietoris-Rips complex $\mathrm{Rips}(S, r)$ two vertices are connected by an edge if and only if the distance between them is at most r. To check whether the points u and v are at a distance at most r, we draw closed balls of radius $\frac{r}{2}$ centered at u and v and check if they intersect. The graph we obtain by finding all pairwise intersections is the 1-skeleton of $\mathrm{Rips}(S, r)$, which we will denote by $\mathrm{Rips}(S, r)^{(1)}$. A subset σ of S will be a simplex of $\mathrm{Rips}(S, r)$ if and only if for all $u, v \in \sigma$ the edge uv is in $\mathrm{Rips}(S, r)^{(1)}$. So, once we have constructed $\mathrm{Rips}(S, r)^{(1)}$ we are left with the problem of finding all possible complete subgraphs (cliques).

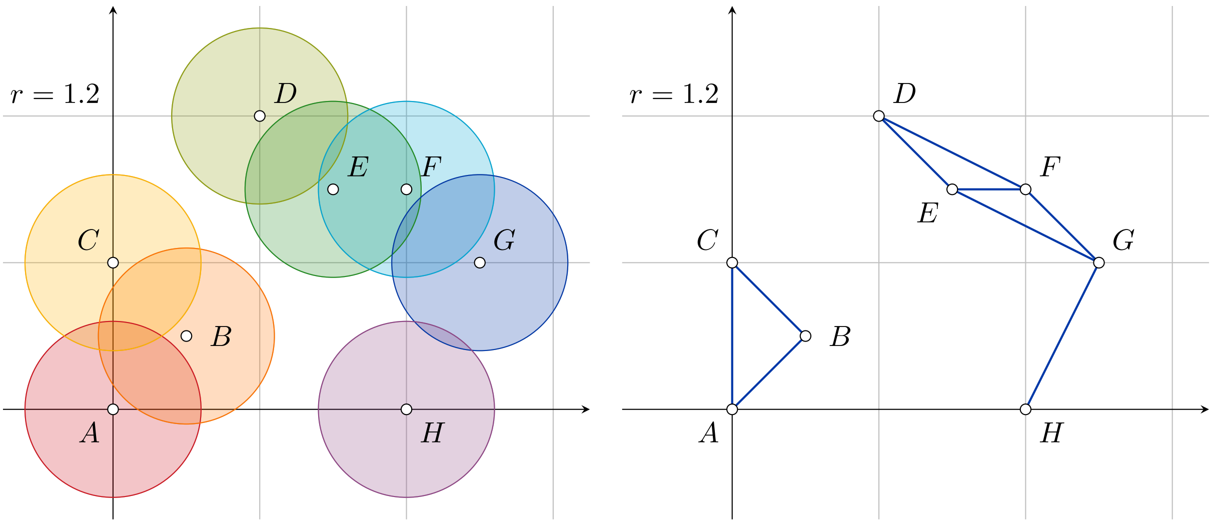




For $r = 1$ we get

$$\mathrm{Rips}(S, 1)^{(1)} = \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,\ AB,\ AC,\ BC,\ DE,\ EF,\ FG\}.$$

There is one 1-cycle in $\mathrm{Rips}(S, 1)^{(1)}$, so we need to add the 2-simplex ABC to $\mathrm{Rips}(S, 1)^{(1)}$ to get $\mathrm{Rips}(S, 1)$. There are no simplices of dimension higher than 2, so $\dim(\mathrm{Rips}(S, 1)) = 2$.



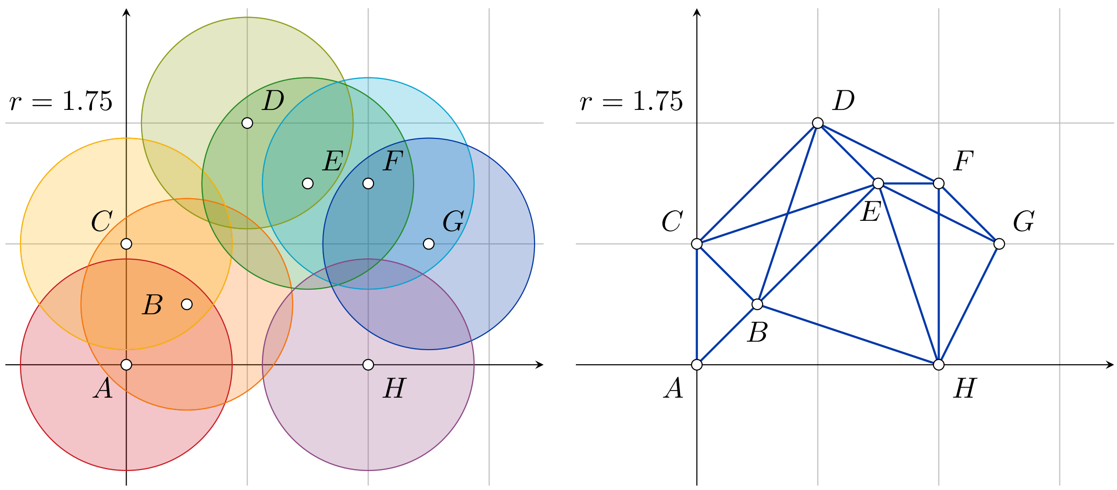
For $r = 1.2$ we get

$\text{Rips}(S, 1.2)^{(1)} = \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,\ AB,\ AC,\ BC,\ DE,\ DF,\ EF,\ EG,\ FG,\ GH\}$.

We have three copies of K_3 (ABC, DEF, EFG) and $\text{Rips}(S, 1.2)^{(1)}$ does not contain K_4, so

$$\text{Rips}(S, 1.2) = \text{Rips}(S, 1.2)^{(1)} \cup \{ABC,\ DEF,\ EFG\}$$

and $\dim(\text{Rips}(S, 1.2)) = 2$.



The 1-skeleton of $\text{Rips}(S, 1.75)$ contains all 8 vertices as well as the edges

$$AB,\ AC,\ BC,\ BD,\ BE,\ BH,\ CD,\ CE,\ DE,\ DF,\ EF,\ EG,\ EH,\ FG,\ FH,\ GH.$$

This graph contains two copies of K_4 ($BCDE$ and $EFGH$), three copies of K_3 that are not contained in K_4 (ABC, BEH, DEF) and no copies of K_5 or higher. In cases like this listing all simplices is not very efficient, so we instead only list those simplices which are not faces of higher-dimensional simplices (we call these maximal simplices) with the understanding that all their faces are also included in the simplicial complex and do not need to be explicitly stated. The maximal simplices of $\text{Rips}(S, 1.75)$ are

$$BCDE,\ EFGH,\ ABC,\ BEH \text{ and } DEF.$$

We also see that $\dim(\text{Rips}(S, 1.75)) = 3$. In this case the area covered by the sensors is connected, which is not true for $r = 1$ and $r = 1.2$. It does however contain a hole because triple intersection of the circles centered at B, E and H is empty (they do intersect pairwise, which is why BEH is a simplex in $\text{Rips}(S, 1.75)$).

Solution 29.

The Čech complex $\mathrm{Cech}(S, r)$ is the nerve of the cover of S with balls of radius r, so we can reuse the images from the previous problem. Also,

$$\mathrm{Cech}(S, r)^{(1)} = \mathrm{Rips}(S, 2r)^{(1)} \quad \text{and} \quad \mathrm{Cech}(S, r) \subseteq \mathrm{Rips}(S, 2r).$$

Since we have already computed the corresponding Vietoris-Rips complexes, we only need to check each $\sigma \in \mathrm{Rips}(S, 2r)$ of dimension higher than 2 to see if it also lies in $\mathrm{Cech}(S, r)$ or not. When $r = 0.5$ we have

$$\mathrm{Rips}(S, 1) = \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,\ AB,\ BC,\ DE,\ EF,\ FG\}$$

and since there are no simplices of dimension 2 or more, we have

$$\mathrm{Cech}(S, 0.5) = \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,\ AB,\ BC,\ DE,\ EF,\ FG\}.$$

When $r = 0.6$ we have

$$\begin{aligned} \mathrm{Rips}(S, 1.2) \;=\; & \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H, \\ & AB,\ AC,\ BC,\ DE,\ DF,\ EF,\ EG,\ FG,\ GH, \\ & ABC,\ DEF,\ EFG\}. \end{aligned}$$

In each case the corresponding three circles have a non-empty triple intersection, so all three 2-simplices from $\mathrm{Rips}(S, 1.2)$ are also contained in $\mathrm{Cech}(S, 0.6)$ and

$$\begin{aligned} \mathrm{Cech}(S, 0.6) \;=\; & \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H, \\ & AB,\ AC,\ BC,\ DE,\ DF,\ EF,\ EG,\ FG,\ GH, \\ & ABC,\ DEF,\ EFG\}. \end{aligned}$$

Finally,

$$\begin{aligned} \mathrm{Rips}(S, 1.75)^{(1)} \;=\; & \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H, \\ & AB,\ AC,\ BC,\ BD,\ BE,\ BH,\ CD,\ CE, \\ & DE,\ DF,\ EF,\ EG,\ EH,\ FG,\ FH,\ GH\} \end{aligned}$$

and the maximal simplices of $\mathrm{Rips}(S, 1.75)$ are

$$BCDE,\ EFGH,\ ABC,\ BEH \text{ and } DEF.$$

The circles centered at B, C, D and E intersect in a quadruple intersection, so $BCDE$ is a maximal simplex of $\mathrm{Cech}(S, 0.875)$. The same is true for $EFGH$. The maximal 2-simplex ABC of $\mathrm{Rips}(S, 1.75)$ corresponds to a triple intersection, so ABC is also a maximal simplex of $\mathrm{Cech}(S, 0.875)$. The same is true for DEF. Now, consider the 2-simplex BEH. It is a maximal simplex in $\mathrm{Rips}(S, 1.75)$, but the triple intersection is empty, so BEH is not a simplex in $\mathrm{Cech}(S, 0.875)$. Instead, $\mathrm{Cech}(S, 0.875)$ has three other maximal simplices, BE, BH and EH. The maximal simplices of $\mathrm{Cech}(S, 0.875)$ are

$$BCDE,\ EFGH,\ ABC,\ DEF,\ BE,\ BH \text{ and } EH.$$

In this case $\mathrm{Cech}(S, 0.875) \neq \mathrm{Rips}(S, 1.75)$.

Solution 30.

We can summarize the distances between each pair of words in the following array:

	SONCE	SENCE	SREDA	VRAG	SRENJ
SONCE	0	1	4	5	4
SENCE	1	0	4	5	3
SREDA	4	4	0	4	2
VRAG	5	5	4	0	4
SRENJ	4	3	2	4	0

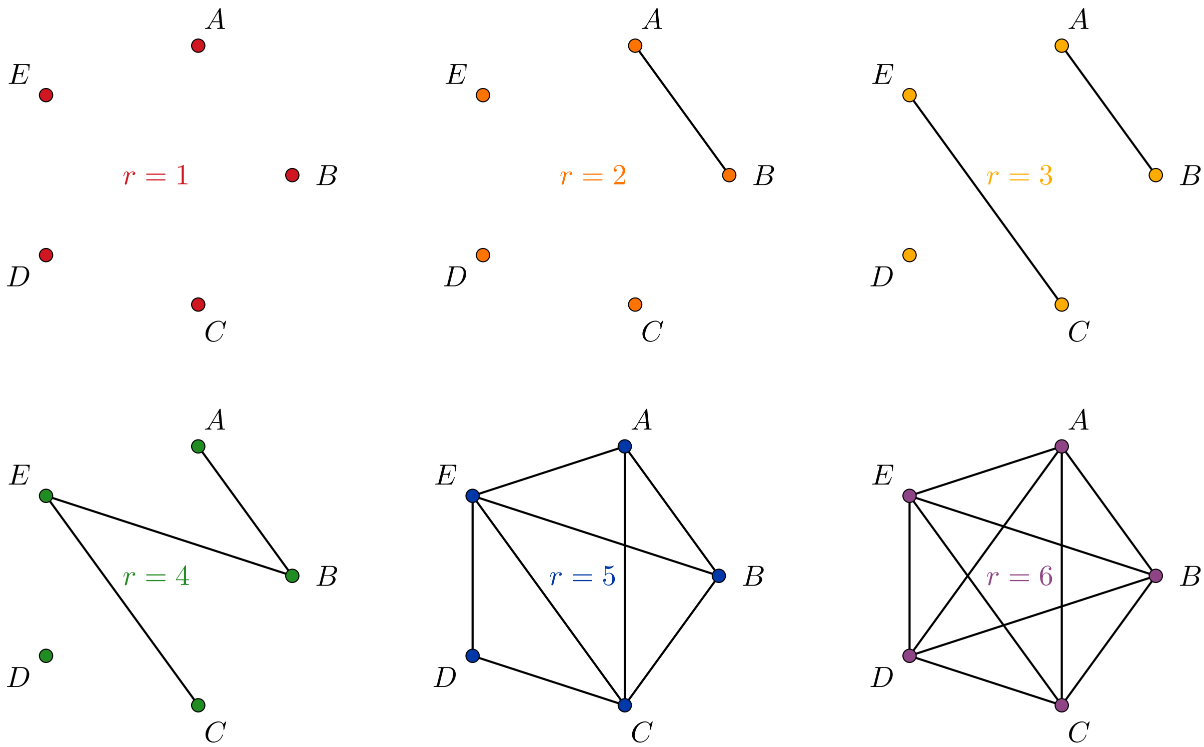
For example, the distance between `SONCE` and `SENCE` is 1 since we only need to replace `O` by `E`. The distance between `SENCE` and `SRENJ` is 3:

$$\texttt{SENCE} \to \texttt{SENJE} \to \texttt{SENJ} \to \texttt{SRENJ}.$$

Let $A = \texttt{SONCE}$, $B = \texttt{SENCE}$, $C = \texttt{SREDA}$, $D = \texttt{VRAG}$, $E = \texttt{SRENJ}$. Then

$$\begin{aligned}
\mathrm{Rips}(S,1)^{(1)} &= \{A,B,C,D,E\},\\
\mathrm{Rips}(S,2)^{(1)} &= \mathrm{Rips}(S,1)^{(1)} \cup \{AB\},\\
\mathrm{Rips}(S,3)^{(1)} &= \mathrm{Rips}(S,2)^{(1)} \cup \{CE\},\\
\mathrm{Rips}(S,4)^{(1)} &= \mathrm{Rips}(S,3)^{(1)} \cup \{BE\},\\
\mathrm{Rips}(S,5)^{(1)} &= \mathrm{Rips}(S,4)^{(1)} \cup \{AC, AE, BC, CD, DE\},\\
\mathrm{Rips}(S,6)^{(1)} &= \mathrm{Rips}(S,5)^{(1)} \cup \{AD, BD\}.
\end{aligned}$$

If we draw the graphs corresponding to the 1-skeletons, it is not difficult to find all maximal complete subgraphs.

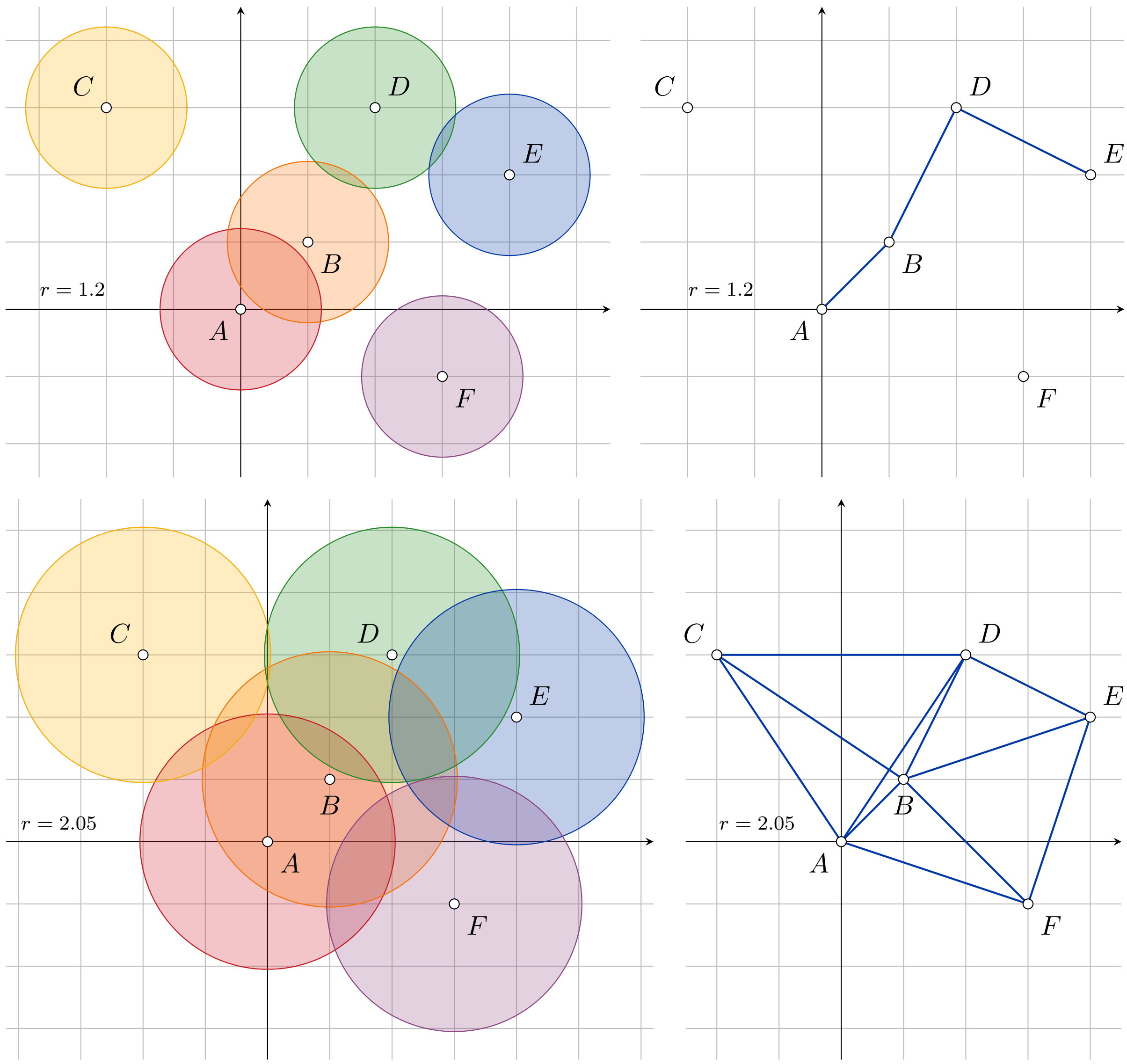


We will list only the maximal simplices for each complex:

$$
\begin{aligned}
\mathrm{Rips}(S,1)\colon &\quad A,\ B,\ C,\ D,\ E;\\
\mathrm{Rips}(S,2)\colon &\quad AB,\ C,\ D,\ E;\\
\mathrm{Rips}(S,3)\colon &\quad AB,\ CE,\ D;\\
\mathrm{Rips}(S,4)\colon &\quad AB,\ BE,\ CE,\ D;\\
\mathrm{Rips}(S,5)\colon &\quad ABCE,\ CDE;\\
\mathrm{Rips}(S,6)\colon &\quad ABCDE.
\end{aligned}
$$

Solution 31.

We need to draw two pictures: one with disks of radius $R/2 = r = 1.2$, the other with disks of radius $R/2 = r = 2.05$.



a. When $R = 2.4$ we get

$$\mathrm{Rips}(S, 2.4) = \{A,\ B,\ C,\ D,\ E,\ F,\ AB,\ BD,\ DE\}.$$

There are no cycles in the 1-skeleton, so there are no triangles or simplices of higher dimensions.

b. When $R = 4.1$ the 1-skeleton $\mathrm{Rips}(S, 4.1)^{(1)}$ contains all vertices as well as the edges

$$AB,\ AC,\ AD,\ AF,\ BC,\ BD,\ BE,\ BF,\ CD,\ DE,\ EF.$$

To get $\text{Rips}(S, 4.1)$ we must add the triangles

$$ABC,\ ABD,\ ABF,\ ACD,\ BCD,\ BDE,\ BEF$$

and the tetrahedron $ABCD$. The 1-skeleton contains no K_5, so there are no simplices of dimension 4 or higher.

c. Since there are no triple intersections of the disks, we have

$$\text{Cech}(S, 1.2) = \text{Cech}(S, 1.2)^{(1)} = \text{Rips}(S, 2.4)^{(1)} = \text{Rips}(S, 2.4).$$

d. We have

$$\text{Cech}(S, 2.05)^{(1)} = \text{Rips}(S, 4.1)^{(1)}.$$

Now we need to check which of the higher dimensional simplices of $\text{Rips}(S, 4.1)$ are contained in $\text{Cech}(S, 4.1)$ by looking at the intersections of the corresponding disks. We see that of the triangles ABC, ABD, ABF, ACD, BCD, BDE, BEF only ACD is not in $\text{Cech}(S, 4.1)$ because the disks centered at A, C and D have an empty triple intersection. We can also conclude that the tetrahedron $ABCD$ will not be contained in $\text{Cech}(S, 4.1)$ because one of its faces is missing.

To summarize, here is the list of maximal simplices for each complex:

$$\begin{aligned}
\text{Cech}(S, 2.4)&: \quad AB,\ BD,\ DE,\ C,\ F;\\
\text{Cech}(S, 4.1)&: \quad ABCD,\ ABF,\ BDF,\ BEF;\\
\text{Cech}(S, 1.2)&: \quad AB,\ BD,\ DE,\ C,\ F;\\
\text{Cech}(S, 2.05)&: \quad ABC,\ ABD,\ ABF,\ BCD,\ BDE,\ BEF.
\end{aligned}$$

5. Simplicial homology

SOLUTION 32.

a. To construct the cone CX we start with the list of all simplices of X, add an extra vertex (we will denote it by $*$) and then add all possible new simplices we can form with the extra vertex. We get

$$A, B, C, AB, AC, BC, *, *A, *B, *C, *AB, *AC, *BC.$$

Once we sort the simplices first by dimension and then lexicographically (assuming * comes before all other characters), we get

$$CX = \{*, A, B, C, *A, *B, *C, AB, AC, BC, *AB, *AC, *BC\}.$$

Similarly, adding all the new simplices to simplices of Y we get

$$A, B, C, D, AB, AD, BC, CD, *, *A, *B, *C, *D, *AB, *AD, *BC, *CD,$$

so

$$CY = \{*, A, B, C, D, *A, *B, *C, *D, AB, AD, BC, CD, *AB, *AD, *BC, *CD\}.$$

b. Consider the simplices of the highest dimension in CX, namely the triangles $*AB$, $*AC$ and $*BC$. The edge AB of $*ABC$ is only contained in the triangle $*AB$, so we can collapse the pair $(*AB, AB)$. The same is true for the pairs $(*AC, AC)$

and $(*BC, BC)$. Once we remove these three pairs of simplices, we can repeat the procedure with the pairs $(*A, A)$, $(*B, B)$, $(*C, C)$. We get a sequence of collapses

$$\begin{aligned} CX &= \{*, A, B, C, *A, *B, *C, AB, AC, BC, *AB, *AC, *BC\} \to \\ &\to \{*, A, B, C, *A, *B, *C, AC, BC, *AC, *BC\} \to \\ &\to \{*, A, B, C, *A, *B, *C, BC, *BC\} \to \\ &\to \{*, A, B, C, *A, *B, *C\} \to \\ &\to \{*, B, C, *B, *C\} \to \\ &\to \{*, C, *C\} \to \\ &\to \{*\}. \end{aligned}$$

We are left with a single vertex $*$, so the complex CX is collapsible. An analoguous sequence of collapses works to show that CY is collapsible.

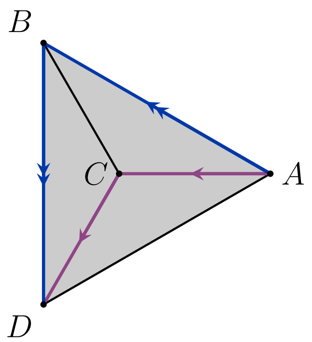
c. The cone CX is collapsible for all simplicial complexes X. To see this, consider any maximal simplex σ of X. Recall that a simplex of X is maximal if it is not a face of a higher dimensional simplex of X. Note that a maximal simplex is not necessarily of dimension $\dim(X)$. Also, at least one maximal simplex exists in any finite simplicial complex. If σ is a maximal simplex of X, then the pair $(*\sigma, \sigma)$ can be collapsed. We get a new complex that has two simplices less than the previous and is equal to $C(X - \sigma)$. Using induction on the number of simplices in X, we can show that repeatedly removing pairs of the form $(*\sigma, \sigma)$ where $*\sigma$ is a maximal simplex at each step will eventually leave us with only $\{*\}$.

Solution 33.

Note that Y is not a simplicial complex since some of the simplices are glued to themselves. We could fix this by taking a sufficiently fine subdivision, but we can do the collapses without it. The simplices of Y are

$$A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, ABCD.$$

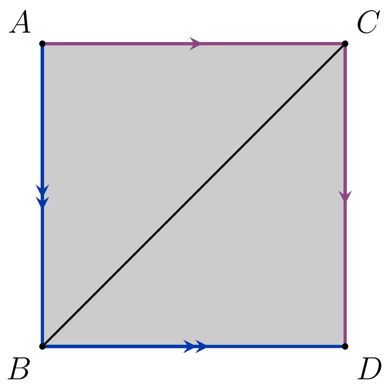
We begin by collapsing the pair $(ABCD, ABD)$. The remaining space can be drawn flat in the plane (as long as we do not try to physically identify the corresponding edges).



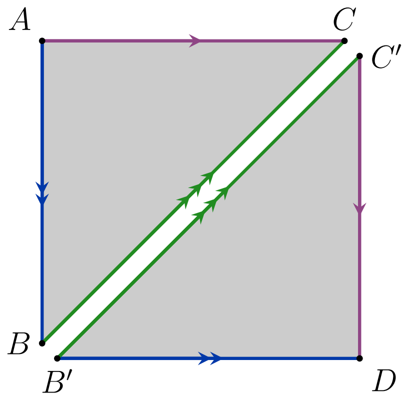
The remaining simplices are

$$A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, ACD, BCD.$$

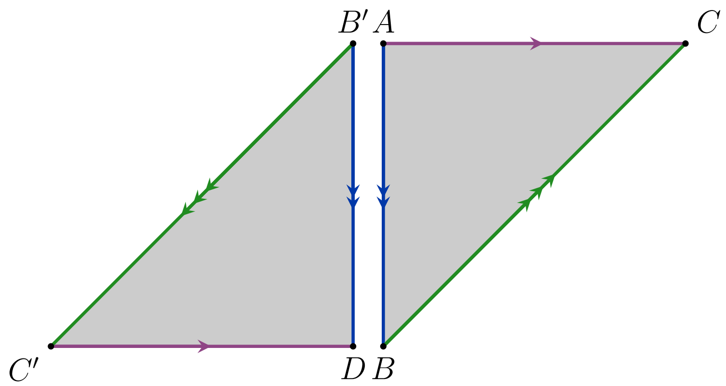
Next, we collapse the pair (ACD, AD).



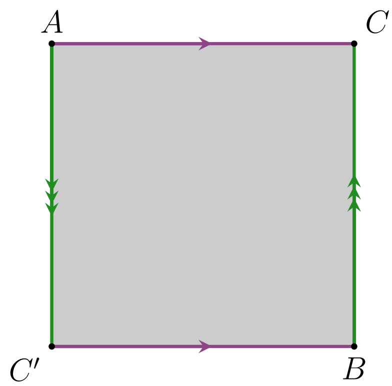
To proceed we temporarily cut the space along the segment BC. We mark both edges so that we will later know how they need to be glued back together.



We can now try to glue together some of the other edges. For example, let us identify the edges AB and BD. While doing so we need to ensure that B' coincides with A and D coincides with B.



Once the two edges are glued together, we no longer need to keep track of them, and we can straighten out the remaining edges to get the space shown in the following figure.



This space is a square with two opposite edges glued together without a twist and the remaining two edges glued together in opposite orientations. We have therefore found a series of collapses and deformations that turns the given space Y into a Klein bottle, so the two spaces are homotopy equivalent.

Solution 34.

Let us begin by listing the simplices of the two complexes:

$$\begin{aligned} X &= \{A,\ B,\ C,\ D,\ E,\ F, \\ &\quad AB,\ AC,\ AD,\ AE,\ BD,\ BE,\ BF,\ CD,\ CE,\ CF,\ DF,\ EF, \\ &\quad ABD,\ ABE,\ ACD,\ BEF,\ CDF,\ CEF\}, \\ Y &= \{A,\ B,\ C,\ D,\ E,\ F, \\ &\quad AB,\ AC,\ AD,\ AE,\ AF,\ BD,\ BE,\ CD,\ CE,\ CF,\ DF,\ EF, \\ &\quad ABD,\ ABE,\ ACD,\ AEF,\ CDF,\ CEF\}. \end{aligned}$$

The horizontal edges are free, so we can collapse all the triangles. This means that in X we can collapse the pairs (ABD, BD), (ACD, AC), (CDF, DF), (CEF, CE), (ABE, AE), (BEF, BF) in any order. What is left is

$$X_1 = \{A,\ B,\ C,\ D,\ E,\ F,\ AB,\ AD,\ BE,\ CD,\ CF,\ EF\}.$$

Since each of the vertices of X_1 occurs in exactly two edges, there are no more free faces to collapse. It is not too difficult to see that X_1 is equivalent to a triangulation of a circle S^1 with 6 vertices and 6 edges. We will calculate the homology groups of S^1 later.

Now, consider Y. We can do collapses in a similar way we did for X starting from the horizontal edges and the result would be similar. But there is more than one way to do the collapses, so let's try something else this time. If we collapse the pair (ABD, BD), we are left with a new free edge, AD. So, we can collapse (ACD, AD) on the next step. After that, we can continue in a similar fashion with (CDF, DF) and (CEF, CF). Finally, we collapse (AEF, AF) and (ABE, AE) to get

$$Y_1 = \{A,\ B,\ C,\ D,\ E,\ F,\ AB,\ AC,\ BE,\ CD,\ CE,\ EF\}.$$

This time, we notice that the vertex D only occurs in one edge, so we can further collapse the pair (CD, D). The same is true for F, so we can also collapse (EF, F). We are left with

$$Y_2 = \{A,\ B,\ C,\ E,\ AB,\ AC,\ BE,\ CE\}.$$

These four vertices and edges also define a triangulation of S^1, so $H_*(X_1) = H_*(Y_2)$, but Y_2 has a smaller number of simplices than X_1, so let us now calculate $H_*(Y_2)$. The two non-trivial chain groups are

$$\begin{aligned}\mathcal{C}_1 &= \langle AB,\, AC,\, BE,\, CE\rangle,\\ \mathcal{C}_0 &= \langle A,\, B,\, C,\, E\rangle.\end{aligned}$$

The boundary homomorphisms ∂_i from the chain

$$\langle 0\rangle \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \langle 0\rangle$$

are defined as follows:

$$\begin{aligned}\partial_1(AB) &= B-A,\\ \partial_1(AC) &= C-A,\\ \partial_1(BE) &= E-B,\\ \partial_1(CE) &= E-C,\\ \partial_0(A) &= 0,\\ \partial_0(B) &= 0,\\ \partial_0(C) &= 0,\\ \partial_0(E) &= 0.\end{aligned}$$

So,

$$\begin{aligned}B_1 &= \mathrm{im}\partial_2 = \langle 0\rangle,\\ B_0 &= \mathrm{im}\partial_1 = \langle B-A,\, C-A,\, E-B,\, E-C\rangle,\\ Z_1 &= \ker\partial_1 = \langle AB+BE-CE-AC\rangle,\\ Z_0 &= \ker\partial_0 = \langle A,\, B,\, C,\, E\rangle.\end{aligned}$$

We can now calculate

$$H_1(Y_2;F) = \frac{Z_1}{B_1} = \frac{\langle AB+BE-CE-AC\rangle}{\langle 0\rangle} = \langle AB+BE-CE-AC\rangle = F$$

and

$$H_0(Y_2;F) = \frac{Z_0}{B_0} = \frac{\langle A,\, B,\, C,\, E\rangle}{\langle B-A,\, C-A,\, E-B,\, E-C\rangle} = \langle A\rangle = F.$$

Since the space Y_2 was obtained by collapsing Y and X_1 is homotopy equivalent to Y_2 (both are S^1), the homotopy groups of the cylinder X and the Moebius strip Y are the same as that of Y_2.

Solution 35.

a. We have $\mathcal{C}_2 = \langle ABD\rangle$, $\mathcal{C}_1 = \langle AB,\, AD,\, BC,\, BD,\, CD\rangle$ and $\mathcal{C}_0 = \langle A,\, B,\, C,\, D\rangle$.
b. The boundary homomorphisms connect the chain groups into a sequence

$$\langle 0\rangle \xrightarrow{\partial_3} \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \langle 0\rangle.$$

We have

$$\begin{aligned}
\partial_2(ABD) &= AB + BD - AD, \\
\partial_1(AB) &= B - A, \\
\partial_1(AD) &= D - A, \\
\partial_1(BC) &= C - B, \\
\partial_1(BD) &= D - B, \\
\partial_1(CD) &= D - C, \\
\partial_0(A) &= 0, \\
\partial_0(B) &= 0, \\
\partial_0(C) &= 0, \\
\partial_0(D) &= 0.
\end{aligned}$$

c. There are no cycles for $n = 2$ because X only has one 2-simplex and we need at least four to form a 2-cycle. So,

$$Z_2 = \ker\partial_2 = \langle 0 \rangle.$$

There are two linearly independent cycles for $n = 1$, namely $AB + BD - AD$ and $BC + CD - BD$, so

$$Z_1 = \ker\partial_1 = \langle AB + BD - AD, BC + CD - BD \rangle.$$

Since ∂_0 maps all vertices to 0, we have

$$Z_0 = \ker\partial_0 = \langle A, B, C, D \rangle.$$

d. Obviously, we have

$$B_2 = \mathrm{im}\partial_3 = \langle 0 \rangle.$$

For $n = 1$ we get

$$B_1 = \mathrm{im}\partial_2 = \langle AB + BD - AD \rangle.$$

For $n = 0$ we have

$$B_0 = \mathrm{im}\partial_1 = \langle B - A,\ D - A,\ C - B,\ D - B,\ D - C \rangle.$$

e. We can now calculate the homology groups $H_n = \frac{Z_n}{B_n}$. We get

$$\begin{aligned}
H_2(X;\mathbb{Z}) &= \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle, \\
H_1(X;\mathbb{Z}) &= \frac{Z_1}{B_1} = \frac{\langle AB + BD - AD,\ BC + CD - BD \rangle}{\langle AB + BD - AD \rangle} = \\
&= \langle BC + CD - BD \rangle = \mathbb{Z}, \\
H_0(X;\mathbb{Z}) &= \frac{Z_0}{B_0} = \frac{\langle A,\ B,\ C,\ D \rangle}{\langle B - A,\ D - A,\ C - B,\ D - B,\ D - C \rangle} = \langle A \rangle = \mathbb{Z}.
\end{aligned}$$

f. The calculation does not change much for $\mathbb{Z}_2$ coefficients. In $\mathbb{Z}_2$ we have $2A = 0$, so $A = -A$, so we can write $\partial_1(AB)$ as $B + A$ instead of $B - A$, and so on. The equality $B + A = 0$ still implies $A = B$, so the elements of B_1 still cancel the elements of Z_0 in the exact same way they did in the case of $\mathbb{Z}$ coefficients. It works the same for other n. We get

$$\begin{aligned}
H_2(X;\mathbb{Z}_2) &= \frac{Z_2}{B_2} = \frac{\langle 0\rangle}{\langle 0\rangle} = \langle 0\rangle,\\
H_1(X;\mathbb{Z}_2) &= \frac{Z_1}{B_1} = \frac{\langle AB+BD+AD,\ BC+CD+BD\rangle}{\langle AB+BD+AD\rangle} =\\
&= \langle BC+CD+BD\rangle = \mathbb{Z}_2,\\
H_0(X;\mathbb{Z}_2) &= \frac{Z_0}{B_0} = \frac{\langle A,B,C,D\rangle}{\langle B+A,\ D+A,\ C+B,\ D+B,\ D+C\rangle} = \langle A\rangle = \mathbb{Z}_2.
\end{aligned}$$

g. The Betti numbers of X are $b_2 = 0$, $b_1 = 1$ and $b_0 = 1$.
h. The Euler characteristic of X is $\chi(X) = 1 - 1 + 0 = 0$.

SOLUTION 36.

a. We have $\mathcal{C}_1 = \langle AB,\ AC,\ AD,\ AE,\ BC,\ DE\rangle$ and $\mathcal{C}_0 = \langle A,\ B,\ C,\ D,\ E\rangle$.
b. The boundary homomorphisms connect the chain groups into a sequence

$$\langle 0\rangle \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \langle 0\rangle.$$

We have

$$\begin{aligned}
\partial_1(AB) &= B - A,\\
\partial_1(AC) &= C - A,\\
\partial_1(AD) &= D - A,\\
\partial_1(AE) &= E - A,\\
\partial_1(BC) &= C - B,\\
\partial_1(DE) &= E - D,\\
\partial_0(A) &= 0,\\
\partial_0(B) &= 0,\\
\partial_0(C) &= 0,\\
\partial_0(D) &= 0,\\
\partial_0(E) &= 0.
\end{aligned}$$

c. There are two linearly independent cycles for $n = 1$, namely $AB + BC - AC$ and $AD + DE - AE$, so

$$Z_1 = \ker\partial_1 = \langle AB + BC - AC,\ AD + DE - AE\rangle.$$

Since ∂_0 maps all vertices to 0, we have

$$Z_0 = \ker\partial_0 = \langle A,\ B,\ C,\ D,\ E\rangle.$$

d. Obviously, we have

$$B_1 = \mathrm{im}\partial_2 = \langle 0\rangle.$$

For $n = 0$ we have

$$B_0 = \mathrm{im}\partial_1 = \langle B - A,\ C - A,\ D - A,\ E - A,\ C - B,\ E - D\rangle.$$

e. We can now calculate the homology groups $H_n = \frac{Z_n}{B_n}$. We get

$$\begin{aligned}
H_1(X;\mathbb{Z}) &= \frac{Z_1}{B_1} = \frac{\langle AB+BC-AC,\ AD+DE-AE\rangle}{\langle 0\rangle} = \\
&= \langle AB+BC-AC,\ AD+DE-AE\rangle = \mathbb{Z}\oplus\mathbb{Z}, \\
H_0(X;\mathbb{Z}) &= \frac{Z_0}{B_0} = \frac{\langle A,\ B,\ C,\ D,\ E\rangle}{\langle B-A,\ C-A,\ D-A,\ E-A,\ C-B,\ E-D\rangle} = \langle A\rangle = \mathbb{Z}.
\end{aligned}$$

f. The calculation is similar for $\mathbb{Z}_2$ coefficients. We get

$$\begin{aligned}
H_1(X;\mathbb{Z}_2) &= \frac{Z_1}{B_1} = \frac{\langle AB+BC+AC,\ AD+DE+AE\rangle}{\langle 0\rangle} = \\
&= \langle AB+BC+AC,\ AD+DE+AE\rangle = \mathbb{Z}_2\oplus\mathbb{Z}_2, \\
H_0(X;\mathbb{Z}_2) &= \frac{Z_0}{B_0} = \frac{\langle A,\ B,\ C,\ D,\ E\rangle}{\langle B+A,\ C+A,\ D+A,\ E+A,\ C+B,\ E+D\rangle} = \langle A\rangle = \mathbb{Z}_2.
\end{aligned}$$

g. The Betti numbers of X are $b_1 = 2$ and $b_0 = 1$.
h. The Euler characteristic of X is $\chi(X) = 1 - 2 = -1$.

Solution 37.

a. We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix},$$

so multiplying from the left with a matrix $P_{i,j}$ (obtained from the identity matrix by swapping the i^{th} and the j^{th} row) swaps the i^{th} and the j^{th} row of the matrix on the right. Also,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c & b \\ d & f & e \\ g & i & h \end{bmatrix},$$

so multiplying with $P_{i,j}$ from the right swaps the i^{th} and the j^{th} column.

b. We get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{bmatrix}$$

and

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} a & b & kc \\ d & e & kf \\ g & h & ki \end{bmatrix}.$$

Let M_i be the matrix obtained from the identity matrix by setting the element at the crossing of the i^{th} row and i^{th} column to k instead of 1. We see that multiplying with M_i from the left multiplies the i^{th} row by k and multiplying from the right multiplies the i^{th} column by k.

c. Finally, let $A_{i,j}$ be the matrix obtained from the identity matrix by changing the element in the i^{th} row and j^{th} column from 0 to k (assuming $i \neq j$). We have

$$\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{bmatrix}$$

and

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & ka+b & c \\ d & kd+e & f \\ g & kg+h & i \end{bmatrix}.$$

We see that multiplying by $A_{i,j}$ from the left has the effect of adding k-times the j^{th} row to the i^{th} row and multiplying by $A_{i,j}$ from the right has the effect of adding k-times the i^{th} column to the j^{th} column.

SOLUTION 38.

a. The chain groups are

$$\begin{aligned} \mathcal{C}_2 &= \langle ABC \rangle, \\ \mathcal{C}_1 &= \langle AB,\ AC,\ AD,\ AE,\ BC,\ DE \rangle, \\ \mathcal{C}_0 &= \langle A,\ B,\ C,\ D,\ E \rangle. \end{aligned}$$

b. The matrices D_2 and D_1 that belong to the maps

$$\partial_2 \colon \mathcal{C}_2 \to \mathcal{C}_1 \quad \text{and} \quad \partial_1 \colon \mathcal{C}_1 \to \mathcal{C}_0$$

are

$$D_2 = \begin{array}{c} \\ AB \\ AC \\ AD \\ AE \\ BC \\ DE \end{array} \begin{array}{c} ABC \\ \left[\begin{array}{c} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{array}\right] \end{array} \quad \text{and} \quad D_1 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccccc} AB & AC & AD & AE & BC & DE \end{array} \\ \left[\begin{array}{cccccc} -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \end{array}.$$

We can also define

$$D_3 = \begin{array}{c} \\ ABC \end{array} \begin{array}{c} 0 \\ \left[\begin{array}{c} 0 \end{array}\right] \end{array} \quad \text{and} \quad D_0 = \begin{array}{c} \\ 0 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

c. To compute

$$H_1(X) = \frac{\ker D_1}{\operatorname{im} D_2} = \frac{Z_1}{B_1},$$

we will do a column reduction on D_1 and take note of the steps so we can repeat them on the rows of D_2. Starting with D_1, we get

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccccc} AB & AC & AD & AE & BC & DE \end{array} \\ \left[\begin{array}{cccccc} -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \end{array} \sim$$

$$\sim \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccccc} -AB & AC & AD & AE & BC & DE \end{array} \\ \left[\begin{array}{cccccc} 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \end{array} \sim$$

$$\sim \begin{array}{cc} & \begin{matrix} -AB & AC{-}AB & AD{-}AB & AE{-}AB & BC & DE \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & \begin{matrix} -AB & AB{-}AC & AD{-}AB & AE{-}AB & BC & DE \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & \begin{matrix} -AC & AB{-}AC & AD{-}AC & AE{-}AC & BC{+}AB{-}AC & DE \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & \begin{matrix} -AC & AB{-}AC & AC{-}AD & AE{-}AC & BC{+}AB{-}AC & DE \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & \begin{matrix} -AD & AB{-}AD & AC{-}AD & AE{-}AD & BC{+}AB{-}AC & DE \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & \begin{matrix} -AD & AB{-}AD & AC{-}AD & AD{-}AE & BC{+}AB{-}AC & DE \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & \begin{matrix} -AE & AB{-}AE & AC{-}AE & AD{-}AE & BC{+}AB{-}AC & DE{+}AD{-}AE \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

The last matrix is the reduced column echelon form of D_1. We will denote it by D_1P, because it is the product of D_1 with the matrix P that encodes the transformations we have described above. Here is the recipe for obtaining it:

- multiply the **first column** by -1 (to ensure the topmost element in the first column is 1),

- add the **first column to the second, third and fourth column** (so that the 1 in the first column will be the only one in its row),
- multiply the **second column** by -1 (to change the topmost element to 1),
- add the **second column to the first, third, fourth and fifth column** (to make this 1 unique in its row),
- multiply the **third column** by -1,
- add the **third column to the first, second and fourth column**,
- multiply the **fourth column** by -1,
- add the **fourth column to the first, second, third and sixth column**.

Now we do the same to the rows of D_2:

$$\begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ AD \\ AE \\ BC \\ DE \end{array} & \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AB \\ AC \\ AD \\ AE \\ BC \\ DE \end{array} & \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AB \\ AC-AB \\ AD-AB \\ AE-AB \\ BC \\ DE \end{array} & \begin{bmatrix} -1 \\ -2 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AB \\ AB-AC \\ AD-AB \\ AE-AB \\ BC \\ DE \end{array} & \begin{bmatrix} -1 \\ 2 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AC \\ AB-AC \\ AD-AC \\ AE-AC \\ AB-AC+BC \\ DE \end{array} & \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AC \\ AB-AC \\ AC-AD \\ AE-AC \\ AB-AC+BC \\ DE \end{array} & \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \end{array} \sim$$

$$\sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AD \\ AB-AD \\ AC-AD \\ AE-AD \\ AB-AC+BC \\ DE \end{array} & \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AD \\ AB-AD \\ AC-AD \\ AD-AE \\ AB-AC+BC \\ DE \end{array} & \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \end{array} \sim \begin{array}{cc} & ABC \\ \begin{array}{c} -AE \\ AB-AE \\ AC-AE \\ AD-AE \\ AB-AC+BC \\ AD-AE+DE \end{array} & \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \end{array}$$

We see that these row transformations changed the basis of the codomain of D_2 to match that of the domain of the reduced column echelon form D_1P of D_1. The matrix we obtained is, in fact, $P^{-1}D_2$. The zero columns of D_1P correspond to the generators of Z_1, namely $AB + BC - AC$ and $AD + DE - AE$. If we look at the corresponding rows of $P^{-1}D_2$, we see that only the row corresponding to $AB + BC - AC$ is non-zero and that means only $AB + BC - AC$ is the generator of B_1. So,

$$H_1(X;\mathbb{Z}) = \frac{Z_1}{B_1} = \frac{\langle AB + BC - AC,\ AD + DE - AE\rangle}{\langle AB + BC - AC\rangle} = \langle AD + DE - AE\rangle = \mathbb{Z}.$$

Note that the matrix $P^{-1}D_2$ is **not** the reduced row echelon form of D_2.
We will calculate the other non-trivial homotopy group, H_0, after the collapses. We will see that the collapses save us quite a bit of work.

d. We could collapse the pairs (ABC, AC), (BC, C) and (AB, B) in this order. (There are, of course, other possible collapses we could make that would lead to the same final result.) We can obtain the corresponding boundary matrices D'_n of this simplified simplicial complex X' by simply removing the corresponding columns and rows from the matrices D_n calculated above. We get

$$D_2' = \begin{array}{c} \\ AD \\ AE \\ DE \end{array}\!\!\begin{array}{c} 0 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\end{array}, \quad D_1' = \begin{array}{c} \\ A \\ D \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} AD & AE & DE \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\end{array} \quad \text{and} \quad D_0 = 0 \begin{array}{c} \begin{array}{ccc} A & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\end{array}.$$

We no longer need D_3' because the simplified complex has no simplices in dimension 2. In this case the reduction to $D_1'P'$ will be much shorter and simpler:

$$\begin{array}{c} \\ A \\ D \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} AD & AE & DE \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\end{array} \sim \begin{array}{c} \\ A \\ D \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} -AD & AE & DE \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\end{array} \sim \begin{array}{c} \\ A \\ D \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} -AD & AE{-}AD & DE \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\end{array} \sim$$

$$\begin{array}{c} \\ A \\ D \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} -AD & AD{-}AE & DE \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\end{array} \sim \begin{array}{c} \\ A \\ D \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} -AE & AD{-}AE & AD{-}AE{+}DE \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}\end{array}.$$

From the single zero column we conclude that $Z_1' = \langle AD + DE - AE\rangle$. This time we don't have to perform the corresponding operations on D_2', because the contents of the zero matrix would not have changed and we already know what the basis of the codomain will be. We know there will be no non-zero rows in the end, so $B_1' = \langle 0\rangle$. We get

$$H_1(X;\mathbb{Z}) = H_1(X';\mathbb{Z}) = \frac{Z_1'}{B_1'} = \frac{\langle AD + DE - AE\rangle}{\langle 0\rangle} = \langle AD + DE - AE\rangle = \mathbb{Z}.$$

Since $\dim(X') = 1$, we know that the only other potentially non-trivial homology group of X' is in dimension 0. We need to calculate

$$H_0(X') = \frac{Z_0'}{B_0'}.$$

This time it makes no sense to do a column reduction on D_0' as that will have no effect. If we simply tried to use all zero columns of D_0' as the generators of Z_0' and the corresponding non-zero rows of D_1' as B_0', we would get

$$\frac{\langle A, D, E\rangle}{\langle A, D, E\rangle} = \langle 0\rangle,$$

which we know is not correct. The reason for this is that the non-zero rows of D_1' are not linearly independent, so we cancelled too many generators of Z_0' by using all the non-zero rows of D_1'. This is true in general. Once we have reduced D_n to its reduced column echelon form D_nP, we need to look at all the rows of $P^{-1}D_{n+1}$ that correspond to the zero columns of D_nP and reduce them a bit further until those that remain non-zero are linearly independent. In other words, after we are done with the column reduction from D_n to D_nP, we need to row-reduce the non-zero rows of $P^{-1}D_{n+1}$. We can encode this in some block matrix Q that is the identity matrix on the block corresponding to the zero rows of $P^{-1}D_{n+1}$ (and non-zero columns of D_nP). Then the domain of D_nPQ^{-1} and the codomain of $QP^{-1}D_{n+1}$ still share the same basis. The number of zero columns of D_nPQ^{-1} is still the same (in fact, the matrix did not change at all because all operations were on zero columns, only the basis of the domain has changed) and the number of non-zero rows of $QP^{-1}D_{n+1}$ is now possibly lower than it was before the reduction. In our example it goes down by 1 because one of the rows is a linear combination

of the other two. This gives us the correct result of $b_0 = 1$.
To sum up, since D'_0 is a zero matrix, we skip the column reduction of D'_0 and only do a row reduction on D'_1 instead. We get

$$\begin{array}{cc} & \begin{matrix} AD & AE & DE \end{matrix} \\ \begin{matrix} A \\ D \\ E \end{matrix} & \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{matrix} AD & AE & DE \end{matrix} \\ \begin{matrix} -A \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{matrix} AD & AE & DE \end{matrix} \\ \begin{matrix} -A \\ A+D \\ E \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \sim$$

$$\begin{array}{cc} & \begin{matrix} AD & AE & DE \end{matrix} \\ \begin{matrix} -A \\ -A-D \\ E \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \sim \begin{array}{cc} & \begin{matrix} AD & AE & DE \end{matrix} \\ \begin{matrix} D \\ -A-D \\ A+D+E \end{matrix} & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

This time we are interested in the non-zero rows of the transformed D'_1. We see that $B'_0 = \langle D, -A-D \rangle$. We did not need to keep track of the transformations because D'_0 is a zero matrix. The zero colums of the transformed D'_0 tell us that $Z'_0 = \langle D,\ -A-D,\ A+D+E \rangle$. We see that

$$H_0(X;\mathbb{Z}) = H_0(X';\mathbb{Z}) = \frac{Z'_0}{B'_0} = \frac{D,\ -A-D,\ A+D+E}{D,\ -A-D} = \langle A+D+E \rangle = \mathbb{Z}.$$

Solution 39.

The chain groups are

$$\begin{aligned} \mathcal{C}_2 &= \langle ABD,\ ABE,\ ACD,\ AEF,\ CDF,\ CEF \rangle, \\ \mathcal{C}_1 &= \langle AB,\ AC,\ AD,\ AE,\ AF,\ BD,\ BE,\ CD,\ CE,\ CF,\ DF,\ EF \rangle, \\ \mathcal{C}_0 &= \langle A,\ B,\ C,\ D,\ E,\ F \rangle. \end{aligned}$$

The boundary matrices are

$$D_3 = \begin{array}{cc} & 0 \\ \begin{matrix} ABD \\ ABE \\ ACD \\ AEF \\ CDF \\ CEF \end{matrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}, \quad D_2 = \begin{array}{cc} & \begin{matrix} ABD & ABE & ACD & AEF & CDF & CEF \end{matrix} \\ \begin{matrix} AB \\ AC \\ AD \\ AE \\ AF \\ BD \\ BE \\ CD \\ CE \\ CF \\ DF \\ EF \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array},$$

$$D_1 = \begin{array}{cc} & \begin{matrix} AB & AC & AD & AE & AF & BD & BE & CD & CE & CF & DF & EF \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array},$$

$$D_0 = \begin{array}{cc} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ 0 & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

After performing column reduction on D_1 and the corresponding row reduction on D_2, we get

$$D_1P = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ G \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

and

$$P^{-1}D_2 = \begin{array}{c} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \begin{array}{c} \begin{array}{cccccc} ABD & ABE & ACD & AEF & CDF & CEF \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Note that all the generators of the columns/rows have changed. We could keep track of the details, but they are not really needed. Next, we need to further row-reduce the part of the matrix $P^{-1}D_2$ that corresponds to the zero-columns of D_1P, which is to say, the last seven rows:

$$P^{-1}D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding matrix has rank 6, so the zero-columns of D_1P will give us seven generators and the non-zero rows of $P^{-1}D_2$ will yield six linearly independent relations. We conclude that $b_1 = 1$.

To compute b_0 we do not need to column-reduce the zero matrix D_0. All that remains is the final row-reduction of the entire D_1 (all rows of D_1 correspond to zero columns of D_0), which shows that the rank of D_1 is equal to 5. We get 6 generators from the zero columns of D_0 and 5 relations from D_1, so $b_0 = 6 - 5 = 1$. This is consistent with the Betti numbers for $M \simeq S^1$ we have found in Problem 34.

Solution 40.

The chain groups are

$$\begin{aligned} \mathcal{C}_2 &= \langle ABC,\, ABD,\, ACD,\, BCD \rangle, \\ \mathcal{C}_1 &= \langle AB,\, AC,\, AD,\, BC,\, BD,\, BE,\, CD,\, CE \rangle, \\ \mathcal{C}_0 &= \langle A,\, B,\, C,\, D,\, E \rangle. \end{aligned}$$

The boundary matrices $D_i \colon C_i \to C_{i-1}$ are

$$D_3 = \begin{array}{c} \\ ABC \\ ABD \\ ACD \\ BCD \end{array} \begin{array}{c} 0 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}, \quad D_2 = \begin{array}{c} \\ AB \\ AC \\ AD \\ BC \\ BD \\ BE \\ CD \\ CE \end{array} \begin{array}{c} \begin{array}{cccc} ABC & ABD & ACD & BCD \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$D_1 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccccccc} AB & AC & AD & BC & BD & BE & CD & CE \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}, \quad D_0 = \begin{array}{c} \\ 0 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

To compute $H_2 = \frac{Z_2}{B_2}$ we need to do column reduction on D_2. The corresponding row reduction on D_3 will be trivial since D_3 is a zero matrix, so $B_2 = \langle 0 \rangle$. We will ignore the generators and only determine the number of trivial columns of the reduced D_2 to get the Betti number b_2. We get

$$D_2 \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is one zero column, so we have one generator of Z_2 and since B_2 is trivial we have $b_2 = 1 - 0 = 1$.

To get $H_1 = \frac{Z_1}{B_1}$ we need to column-reduce D_1 and perform the same operations on the rows of D_2. We will once again ignore the generators and focus only on finding the Betti number b_1. We get

$$
\begin{aligned}
D_1 \sim & \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \sim \\
\sim & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 \quad &\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} -1 & -1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & -1 & 1 \\ 2 & 2 & -1 & -1 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 1 & -1 & 1 \\ 2 & 2 & -1 & -1 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & -2 & -1 & 0 \\ -1 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & -2 & -1 & 0 \\ -1 & 1 & 3 & 1 \\ -2 & -1 & 1 & 0 \end{bmatrix}.
\end{aligned}
$$

We now need to further reduce the rows 4, 5, 7 and 8 of the reduced D_2 that correspond to the four zero columns of the reduced D_1. We are only interested in the rank of this matrix. We get

$$
\begin{aligned}
\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ -2 & -1 & 1 & 0 \end{bmatrix}
&\sim \begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 3 & 1 & -1 \\ 0 & 4 & 4 & 0 \\ -1 & 2 & 2 & -1 \end{bmatrix}
\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ -1 & 2 & 2 & -1 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}
\sim \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 3 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}
\sim \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}
\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \\
&\sim \begin{bmatrix} \underline{\mathbf{1}} & 0 & 0 & 1 \\ 0 & \underline{\mathbf{1}} & 0 & -1 \\ 0 & 0 & \underline{\mathbf{1}} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

We have chosen to

- add the second row to the other three,
- divide the first and the third row by 4,
- subtract two times the third row from the fourth and multiply the result by -1,
- subtract the fourth row from the first two,
- subtract the third row from the first and multiply the result by -1,
- subtract three times the third row from the second and divide the result by -2,
- subtract the second row from the first,
- sort the rows in the reverse order and
- subtract the third row from the second.

Of course, this is not the only possible reduction, but it shows that the rank of this matrix is 3, so three of the four rows of the reduced D_2 were linearly independent. We conclude that $b_1 = 4 - 3 = 1$.

Finally, to get H_0, we do not have to do any column operations on the zero matrix D_0, so all that is left is to further row-reduce the entire D_1, because all rows of D_1 correspond to zero columns of D_0. We get

$$D_1 \sim \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} \underline{1} & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \underline{1} & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have chosen to

- multiply the first row by -1,
- subtract the first row from the second and multiply the result by -1,
- subtract the second row from the third and multiply the result by -1,
- subtract the third row from the fourth and multiply the result by -1,
- subtract the fourth row from the fifth.

It is clear that the rank of this matrix is 4 and since D_0 had 5 trivial columns we get $b_0 = 5 - 4 = 1$. To summarize, we have $b_2 = b_1 = b_0 = 1$. Our space has one generator of homology in dimension 2 (the boundary of the tetrahedron $ABCD$), one generator of homology in dimension 1 (the boundary of the triangle BEC) and one generator of homology in dimension 0 (one connected component).

Solution 41.

The chain groups are

$$\begin{aligned}
\mathcal{C}_2 &= \langle ABD,\ ABE,\ ACD,\ ACF,\ AEF,\ BCE,\ BCF,\ BDF,\ CDE,\ DEF\rangle,\\
\mathcal{C}_1 &= \langle AB,\ AC,\ AD,\ AE,\ AF,\ BC,\ BD,\ BE,\ BF,\ CD,\ CE,\ CF,\ DE,\ DF,\ EF\rangle,\\
\mathcal{C}_0 &= \langle A,\ B,\ C,\ D,\ E,\ F\rangle.
\end{aligned}$$

The boundary matrices $D_i\colon C_i \to C_{i-1}$ are

$$D_3 = \begin{array}{cc}
 & 0 \\
\begin{array}{c} ABD\\ ABE\\ ACD\\ ACF\\ AEF\\ BCE\\ BCF\\ BDF\\ CDE\\ DEF \end{array} &
\left[\begin{array}{c} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{array}\right]
\end{array},$$

$$D_2 = \begin{array}{cc}
 & \begin{array}{cccccccccc} ABD & ABE & ACD & ACF & AEF & BCE & BCF & BDF & CDE & DEF \end{array} \\
\begin{array}{c} AB\\ AC\\ AD\\ AE\\ AF\\ BC\\ BD\\ BE\\ BF\\ CD\\ CE\\ CF\\ DE\\ DF\\ EF \end{array} &
\left[\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array},$$

$$D_1 = \begin{array}{cc}
 & \begin{array}{ccccccccccccccc} AB & AC & AD & AE & AF & BC & BD & BE & BF & CD & CE & CF & DE & DF & EF \end{array} \\
\begin{array}{c} A\\ B\\ C\\ D\\ E\\ F \end{array} &
\left[\begin{array}{ccccccccccccccc}
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}\right]
\end{array},$$

$$D_0 = \begin{array}{cc}
 & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
0 & \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{array}.$$

The homology of the real projective plane $P = \mathbb{R}P^2$ with integer coefficients is

$$H_2(P;\mathbb{Z}) = 0, \quad H_1(P;\mathbb{Z}) = \mathbb{Z}_2, \quad H_0(P;\mathbb{Z}) = \mathbb{Z}.$$

The Betti numbers are $b_2 = 0$, $b_1 = 0$ and $b_0 = 1$. The Euler characteristic of the projective plane is $\chi(P) = 1 - 0 + 0 = 1$. The 0 in dimension 2 tells us that P is not orientable. With $\mathbb{Z}_2$ coefficients we get the fundamental class as the generator of the top-dimensional homology group, so

$$H_2(P;\mathbb{Z}_2) = \mathbb{Z}_2, \quad H_1(P;\mathbb{Z}_2) = \mathbb{Z}_2, \quad H_0(P;\mathbb{Z}_2) = \mathbb{Z}_2.$$

You can find the python code to compute the simplicial homology with $\mathbb{Q}$ coefficients here. With some minimal changes to the code you can instead compute the simplicial homology with $\mathbb{Z}_2$ coefficients. Try to change the code to reproduce the results listed above. You only need to make sure that every time two numbers are added the result is reduced modulo 2 before continuing. To implement the homology with $\mathbb{Z}_p$ coefficients you would also need to make sure that all instances where the original algorithm divides a row or a column by the pivot the $\mathbb{Z}_p$ algorithm multiplies it with the multiplicative inverse of the pivot in $\mathbb{Z}_p$.

Solution 42.

The Betti numbers of the torus $T = S^1 \times S^1$ are $b_2 = 1$, $b_1 = 2$ and $b_0 = 1$ and its Euler characteristic is

$$\chi(T) = b_0 - b_1 + b_2 = 1 - 2 + 1 = 0.$$

The homology of the torus with integer coefficients is

$$H_2(T;\mathbb{Z}) = \mathbb{Z}, \quad H_1(T;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}, \quad H_0(T;\mathbb{Z}) = \mathbb{Z}.$$

The homology of the torus with $\mathbb{Z}_2$ coefficients is

$$H_2(T;\mathbb{Z}_2) = \mathbb{Z}_2, \quad H_1(T;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_0(T;\mathbb{Z}_2) = \mathbb{Z}_2.$$

Solution 43.

The homology of the Klein bottle K with integer coefficients is

$$H_2(K;\mathbb{Z}) = 0, \quad H_1(K;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2, \quad H_0(T;\mathbb{Z}) = \mathbb{Z}.$$

The Betti numbers are $b_2 = 0$, $b_1 = 1$ and $b_0 = 1$, so the Euler characteristic is $\chi(K) = 1 - 1 + 0 = 0$. The 0 in dimension 2 tells us that K is not orientable. The homology with $\mathbb{Z}_2$ coefficients is

$$H_2(K;\mathbb{Z}_2) = \mathbb{Z}_2, \quad H_1(K;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_0(T;\mathbb{Z}_2) = \mathbb{Z}_2.$$

6. Persistence

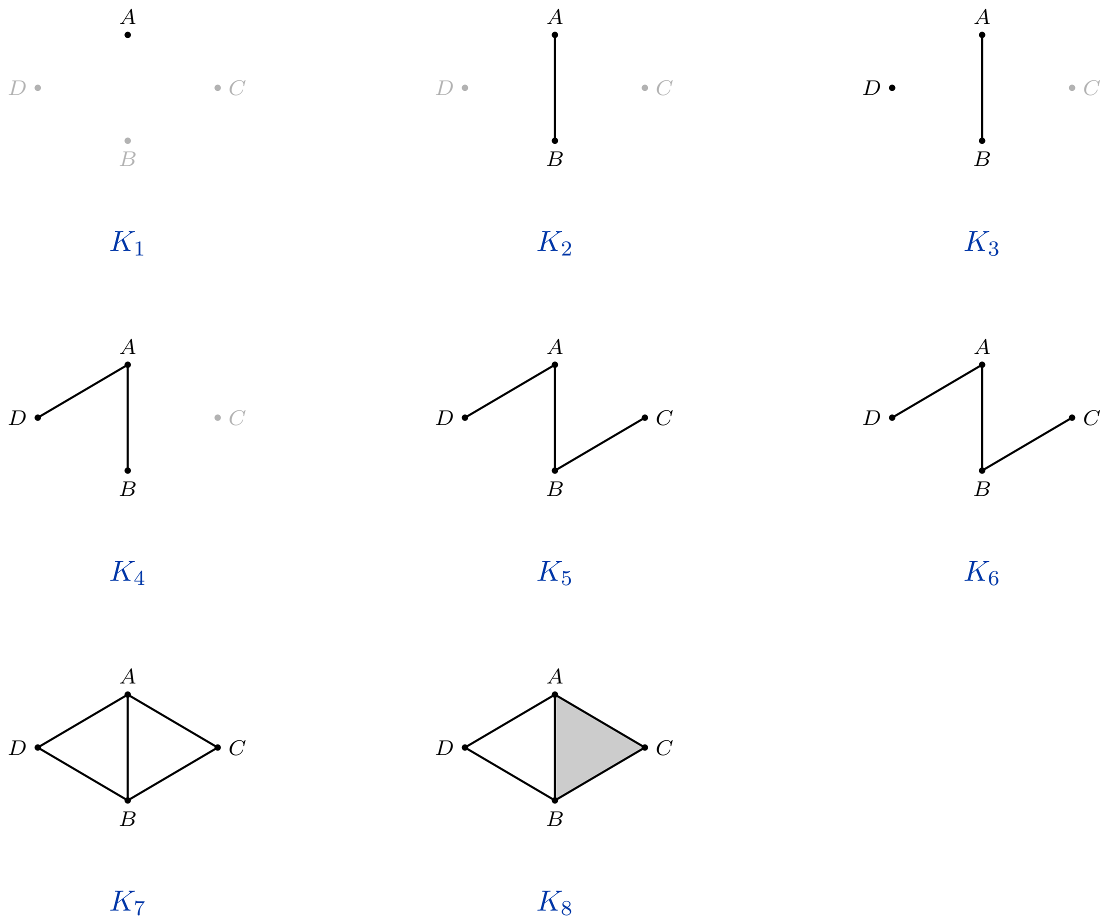
Solution 44.

a. The function f maps as follows:

A	B	C	D	AB	AC	AD	BC	BD	ABC
1	3	5	3	2	7	4	5	7	8

The following table shows at what level each of the simplices is added. Note that B gets added before level 3 suggested by the value of the function. Since a higher-dimensional simplex AB that has B as a face is already added at level 2, the face B needs to be added at that level as well. This is necessary to ensure that at each level the union of all simplices added by then is a subcomplex.

Level	Simplices added at this level
1	A
2	B, AB
3	D
4	AD
5	C, BC
6	
7	AC, BD
8	ABC



K_1 K_2 K_3 K_4 K_5 K_6 K_7 K_8

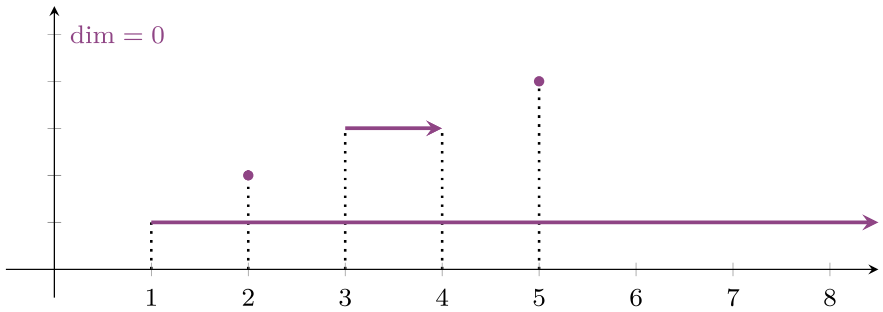
b. In dimension 0
- a class is born at level 1 when A is added,
- a class is born at level 2 when B is added and dies immediately,
- a class is born at level 3 with the addition of D,
- the class born at level 3 dies when we add AD at level 4,
- a class is born at level 5 with the addition of C and dies immediately.

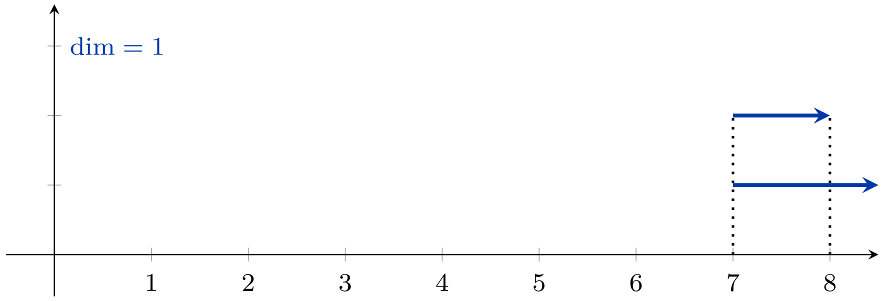
Therefore, the classes in dimension 0 are $[1, \infty)$, $[2, 2)$, $[3, 4)$ and $[5, 5)$.

In dimension 1
- a class is born when AC is added at level 7,
- a class is born when BD is added at level 7,
- the class born at level 7 dies when we add ABC at level 8.

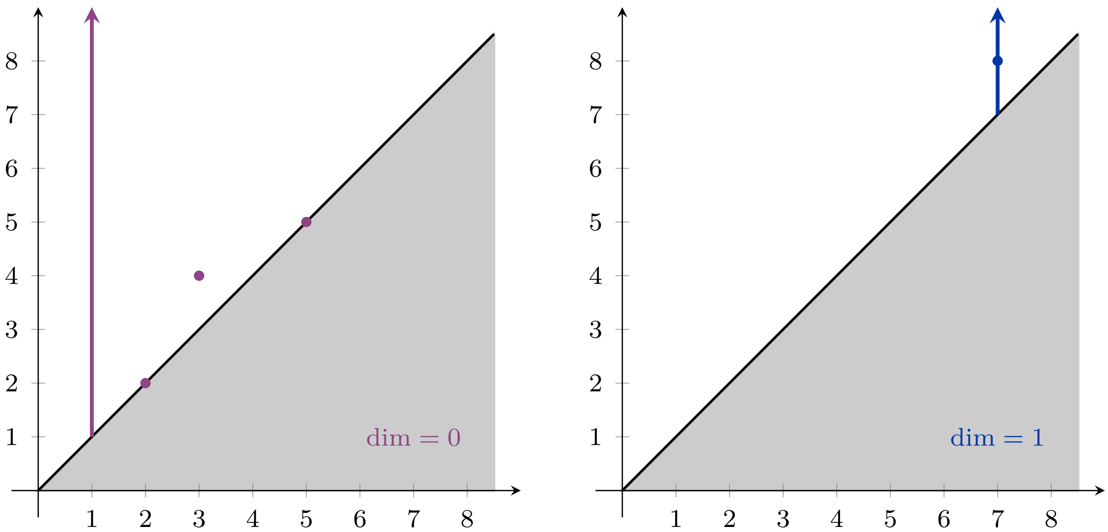
The classes in dimension 1 are $[7, \infty)$ and $[7, 8)$.

c.



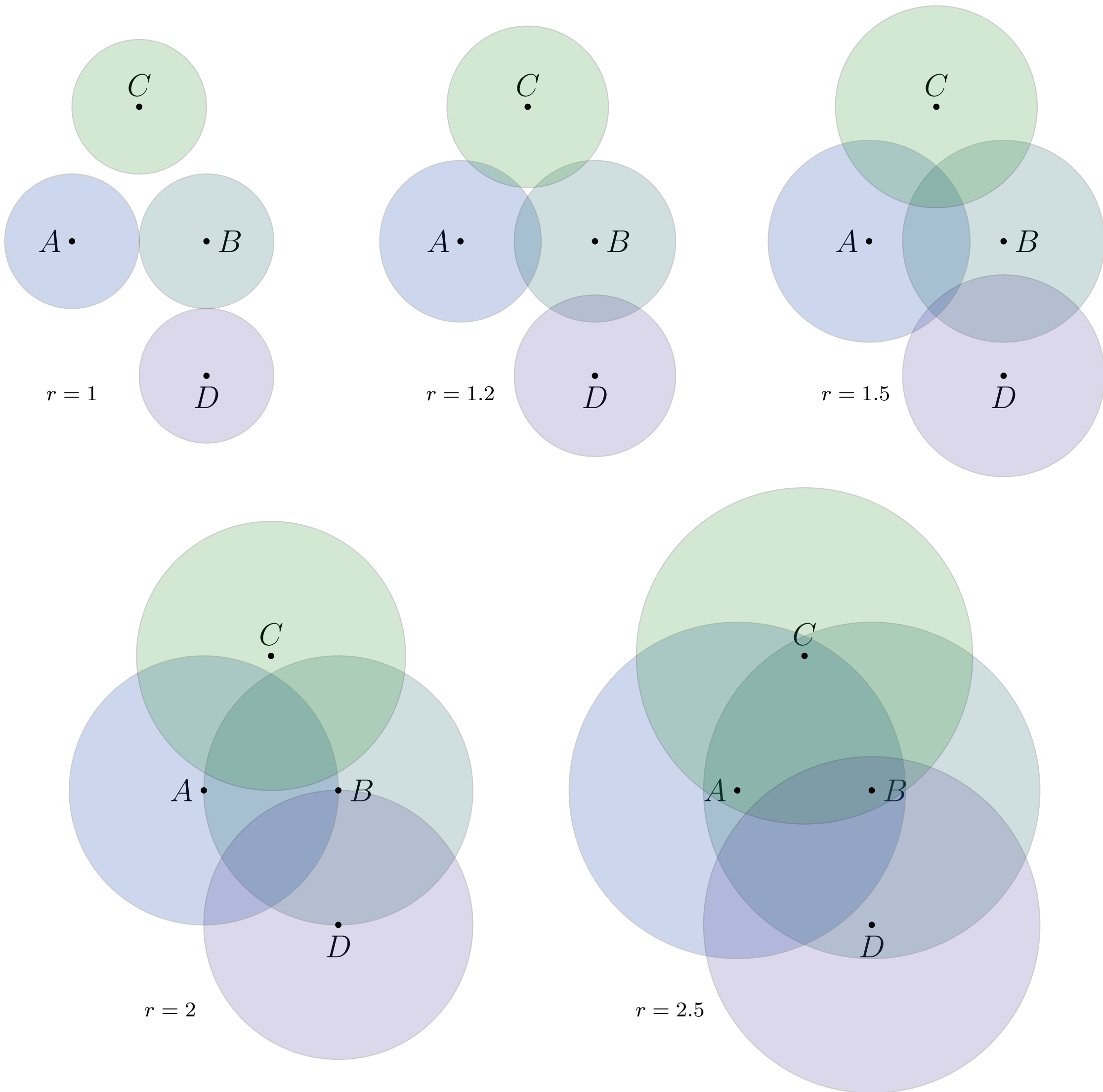


d.



SOLUTION 45.

a. The stages are shown in the following figures.



The corresponding Čech complexes are

$$\begin{aligned}
\mathrm{Cech}(S,0) &= \{A,\ B,\ C,\ D\},\\
\mathrm{Cech}(S,1) &= \mathrm{Cech}(S,0)\ \cup\ \{AB,\ BD\},\\
\mathrm{Cech}(S,1.2) &= \mathrm{Cech}(S,1)\ \cup\ \{AC,\ BC\},\\
\mathrm{Cech}(S,1.5) &= \mathrm{Cech}(S,1.2)\ \cup\ \{AD,\ ABC,\ ABD\},\\
\mathrm{Cech}(S,2) &= \mathrm{Cech}(S,1.5)\ \cup\ \{\},\\
\mathrm{Cech}(S,2.5) &= \mathrm{Cech}(S,2)\ \cup\ \{CD,\ ACD,\ BCD,\ ABCD\}.
\end{aligned}$$

At $r = 0$ we add four classes in dimension 0. When $r = 1$, we add the edges AB and BD, so two of the classes added at $r = 0$ die. At $r = 1.2$ we add two more edges and the complex becomes connected, so one more class born at $r = 0$ dies. The remaining class in dimension 0 survives indefinitely. To sum up, the classes in dimension 0 are $[0, \infty)$, $[0,\ 1.2)$, $[0,\ 1)$ and $[0,\ 1)$.
In dimension 1 the first class, namely the cycle $AB+BC-AC$, is born at $r = 1.2$. This class dies at $r = 1.5$ when the triangle ABC is added. There is also a class corresponding to the boundary of the triangle ABD which is born and dies at $r = 1.5$. Two more 1-dimensional classes (corresponding to the boundaries of the triangles ACD and BCD) are born and die immediately at $r = 2.5$. In any case

these two are not linearly independent, since

$$\partial(ACD) = CD + (AC - AD) = CD + (BC - BD) = \partial(BCD).$$

Here, we used the fact that

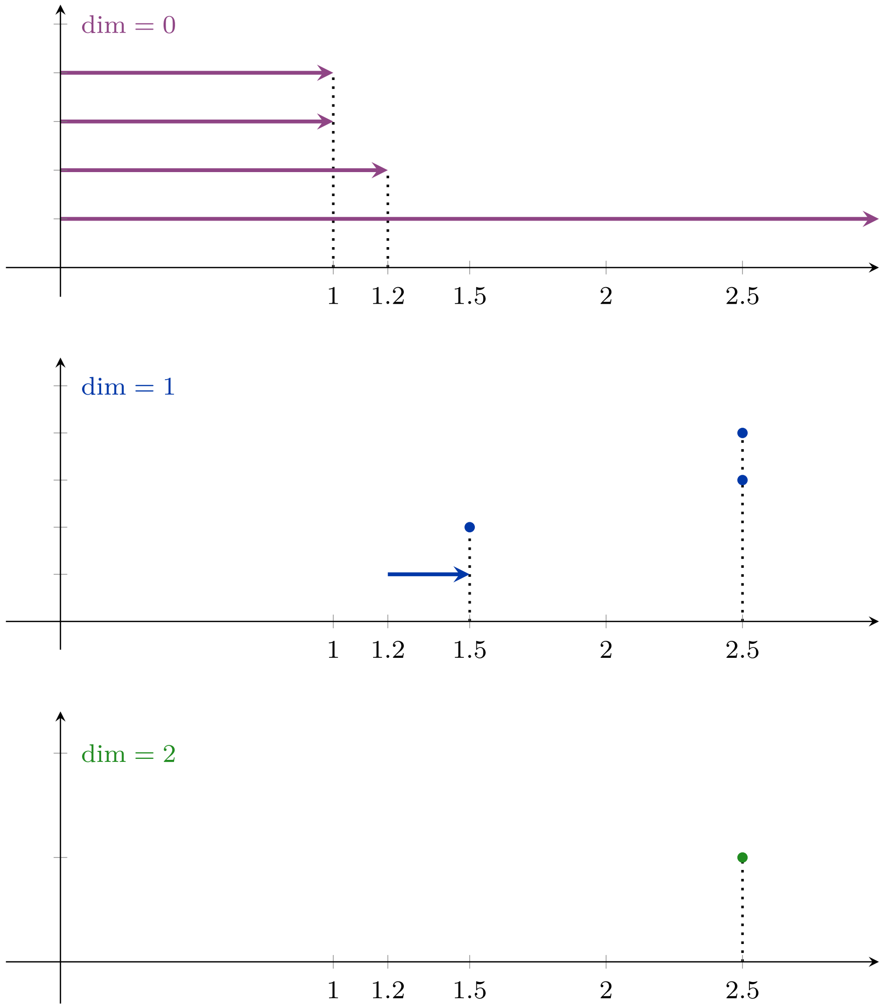
$$\partial(ABC) = AB + BC - AC = 0 \quad \text{and} \quad \partial(ABD) = AB + BD - AD = 0,$$

so

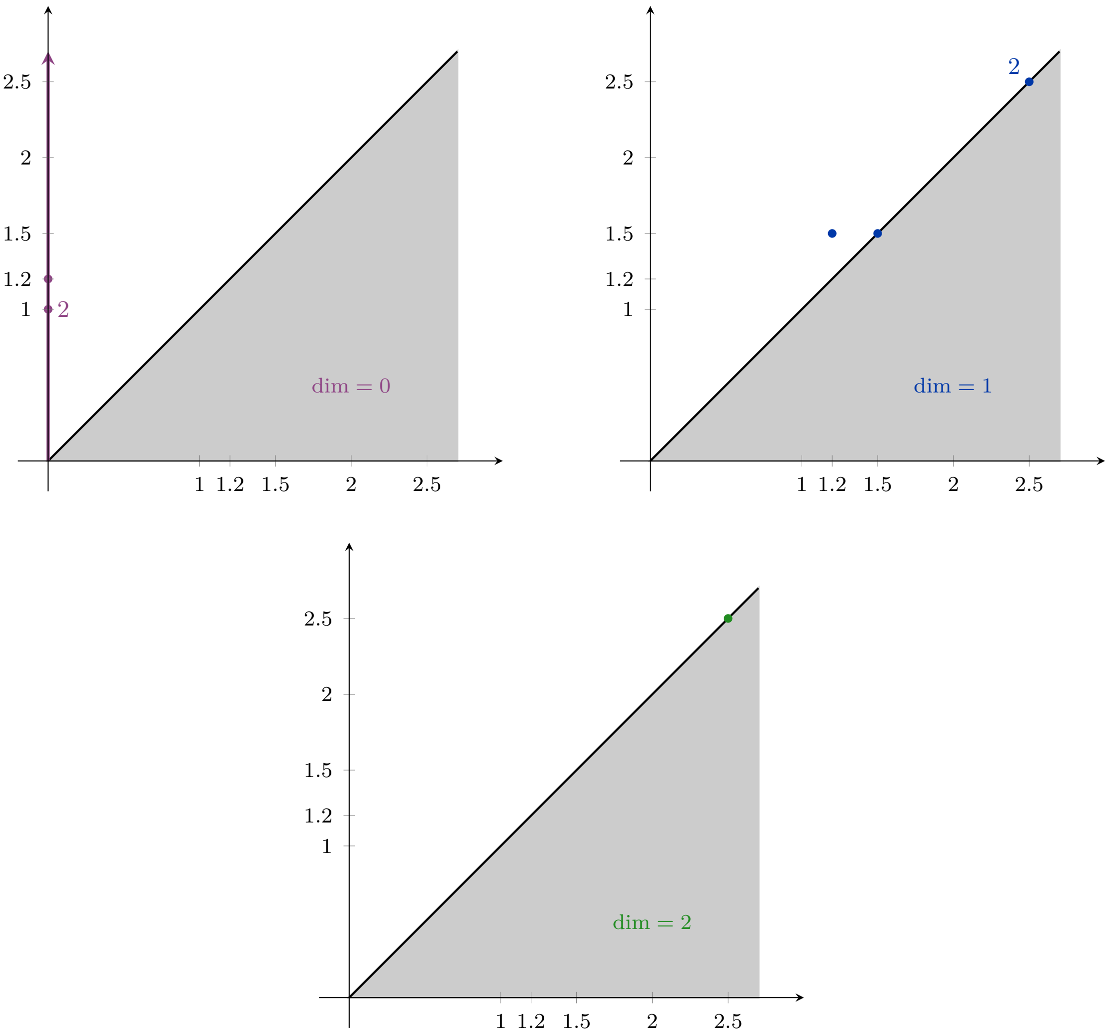
$$\partial(ABD) - \partial(ABC) = BD - AD - BC + AC = 0$$

and thus $AC - AD = BC - BD$. We will later learn that the classes that are born and die at the same time do not have any real importance in the persistence diagram. We will also not need to worry about checking for linear independance since the matrix reduction procedure we will introduce will take care of if for us. The classes in dimension 1 are $[1.2, 1.5)$, $[1.5, 1.5)$, $[2.5, 2.5)$ and $[2.5, 2.5)$.
In dimension 2 a single class is born at $r = 2.5$ and it dies at the same time when we add the 3-simplex $ABCD$. So, the only class in dimension 2 is $[2.5, 2.5)$.

b.



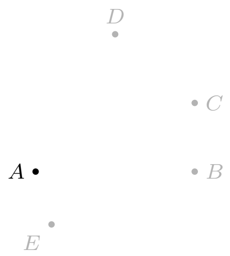
c.



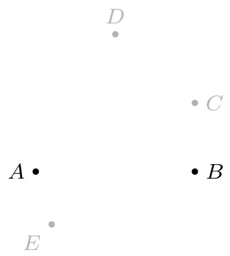
SOLUTION 46.

a. The filtration is

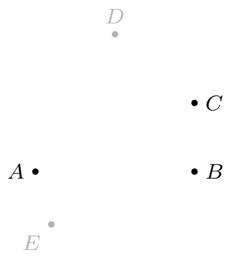
$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
K_0 & = & \{\}, & K_6 & = & K_5 \cup \{AB\}, & K_{12} & = & K_{11} \cup \{BE\}, \\
K_1 & = & K_0 \cup \{A\}, & K_7 & = & K_6 \cup \{AC\}, & K_{13} & = & K_{12} \cup \{CD\}, \\
K_2 & = & K_1 \cup \{B\}, & K_8 & = & K_7 \cup \{AD\}, & K_{14} & = & K_{13} \cup \{ABC\}, \\
K_3 & = & K_2 \cup \{C\}, & K_9 & = & K_8 \cup \{AE\}, & K_{15} & = & K_{14} \cup \{ABD\}, \\
K_4 & = & K_3 \cup \{D\}, & K_{10} & = & K_9 \cup \{BC\}, & K_{16} & = & K_{15} \cup \{ACD\}, \\
K_5 & = & K_4 \cup \{E\}, & K_{11} & = & K_{10} \cup \{BD\}, & K_{17} & = & K_{16} \cup \{BCD\}.
\end{array}
$$



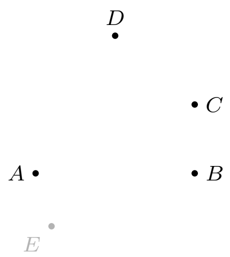
K_1



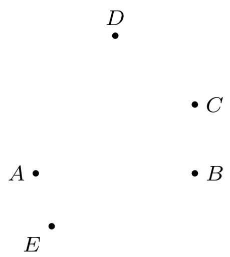
K_2



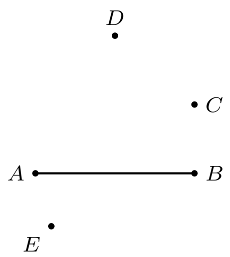
K_3



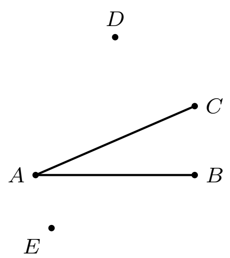
K_4



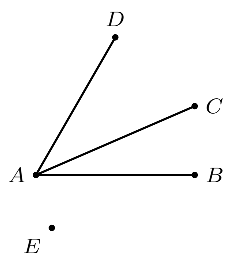
K_5



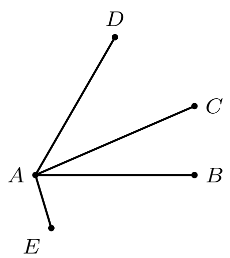
K_6



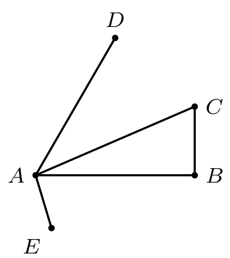
K_7



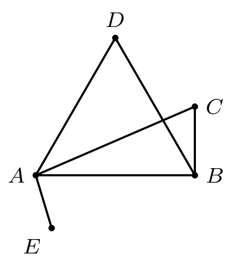
K_8



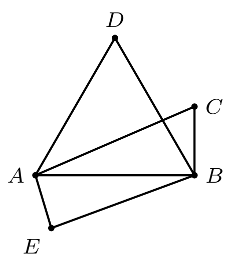
K_9



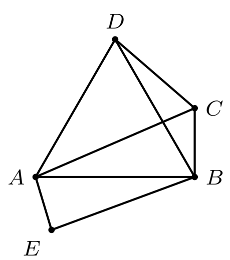
K_{10}



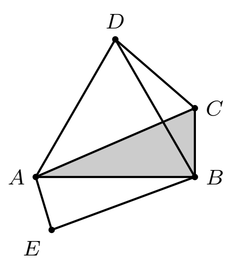
K_{11}



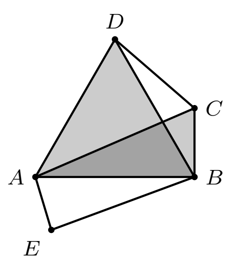
K_{12}



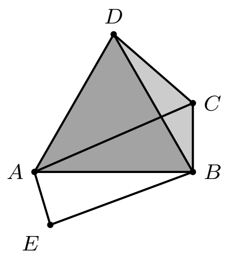
K_{13}



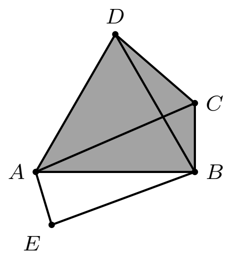
K_{14}



K_{15}

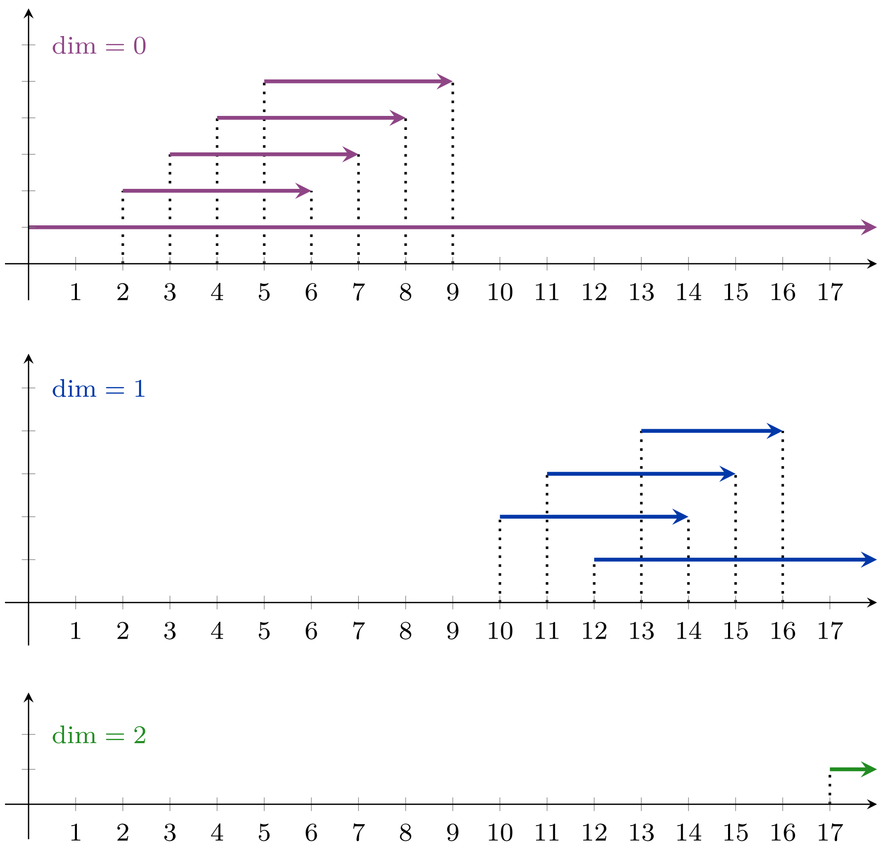


K_{16}

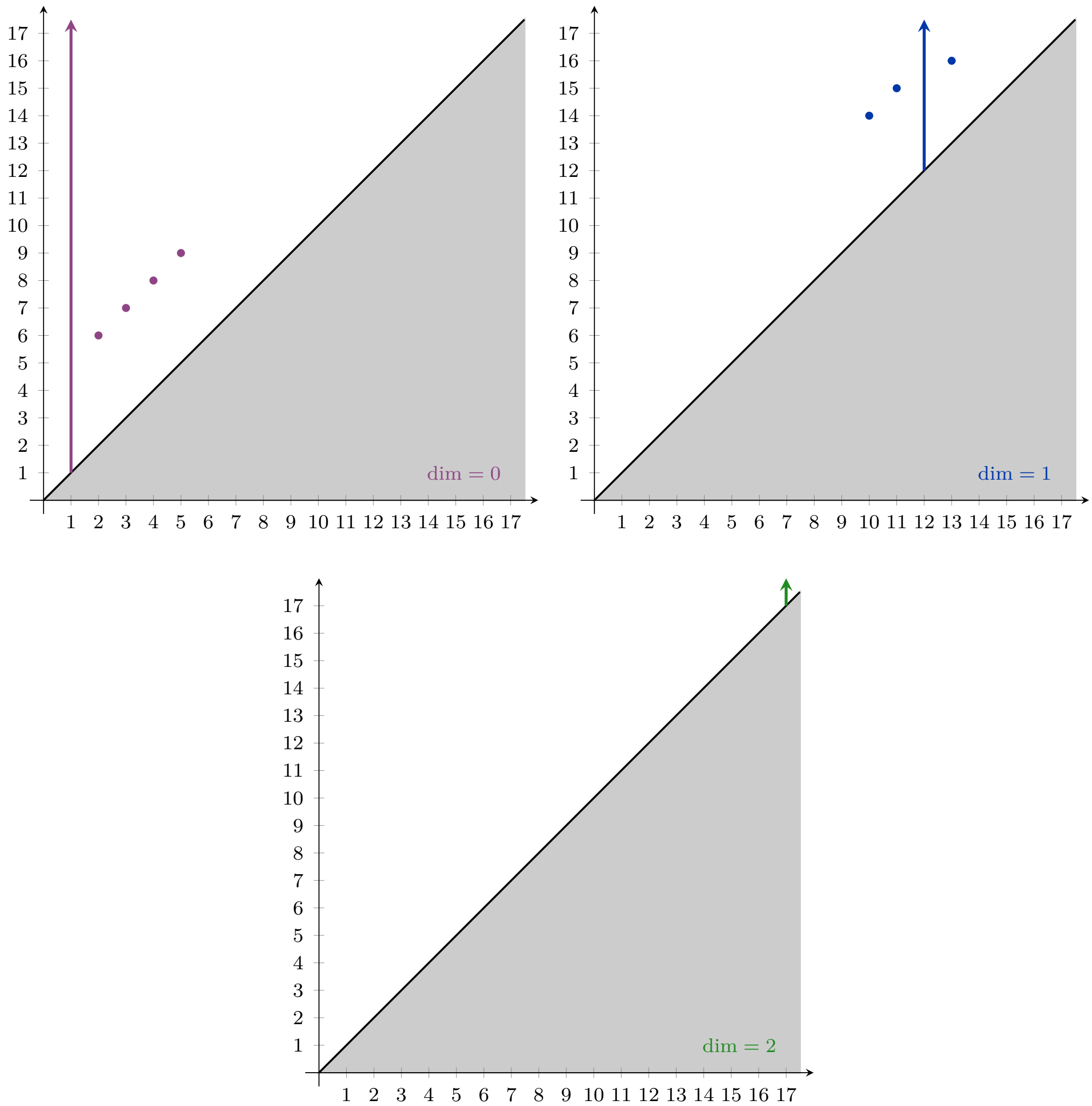


$K_{17} = K$

b. In dimension 0 we have classes born at stages 1 (with the addition of A), 2 (adding B), 3 (adding C), 4 (adding D) and 5 (adding E). Four of them die at stages 6 (adding AB), 7 (adding AC), 8 (adding AD) and 9 (adding AE). Following the elder rule, we get classes $[1, \infty)$, [2, 6), [3, 7), [4, 8) and [5, 9).
In dimension 1 we have classes born at stages 10 ($AB+BC-AC$), 11 ($AB+BD-AD$), 12 ($AB+BE-AE$) and 13 ($AC+CD-AD$; note that $BC+CD-BD$ is a linear combination of the other 3-cycles, so we should not add it). We add the triangles ABC, ABD, and ACD at stages 14, 15 and 16, which is when three of these classes die. Note that adding BCD at 17 does not kill anything because its boundary was already trivial at 16 due to its linear dependence on the other three finite classes. We get [10, 14), [11, 15), [13, 16) and $[12, \infty)$.
In dimension 2 we only have the class $-ABC + ABD - ACD + BCD$ born at stage 17 which never dies, so $[17, \infty)$. The corresponding barcodes are drawn in the following figure.



c.

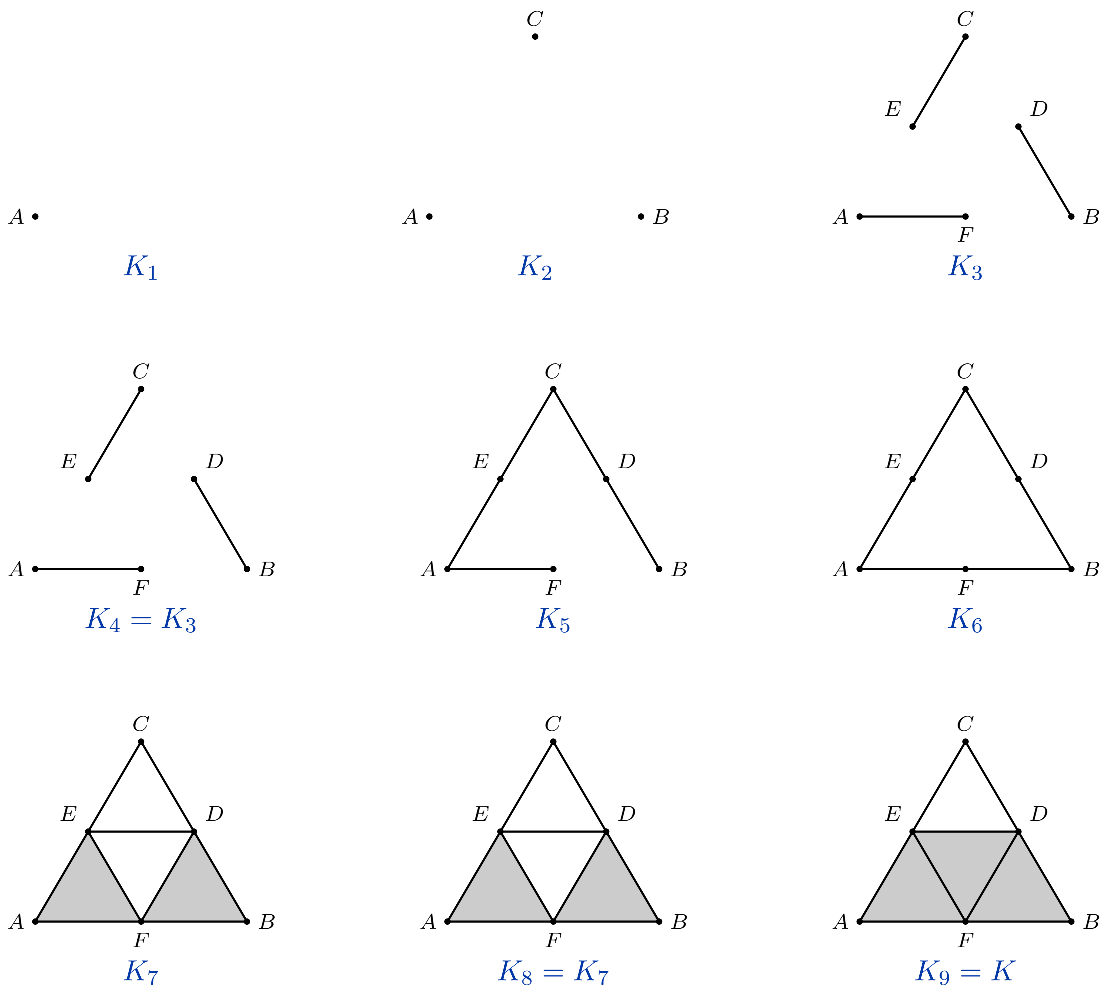


SOLUTION 47.

a. Starting with $K_0 = \{\}$, we have

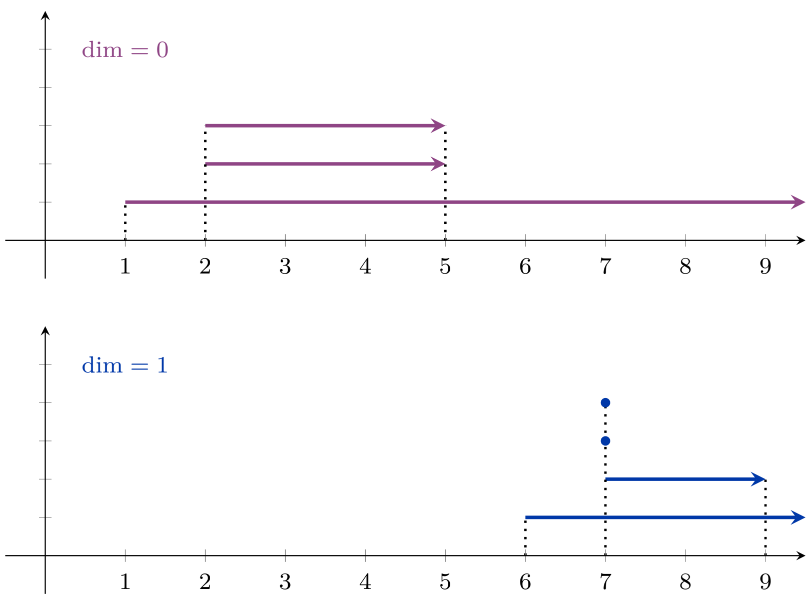
$$\begin{aligned}
K_1 &= K_0 \cup \{A\},\\
K_2 &= K_1 \cup \{B, C\},\\
K_3 &= K_2 \cup \{D, E, F, AF, BD, CE\},\\
K_4 &= K_3 \cup \{\},\\
K_5 &= K_4 \cup \{AE, CD\},\\
K_6 &= K_5 \cup \{BF\},\\
K_7 &= K_6 \cup \{DE, DF, EF, AEF, BDF, \},\\
K_8 &= K_7 \cup \{\},\\
K_9 &= K_8 \cup \{DEF\}.
\end{aligned}$$

The filtration subcomplexes are shown in the following figure.

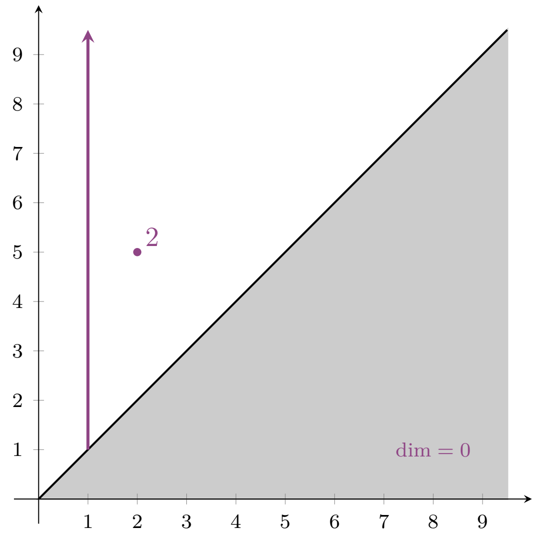


b. In dimension 0 one class is born at stage 1 and two classes are born at stage 2. Both of those die at stage 5. So, we have classes $[1, \infty)$, $[2, 5)$ and $[2, 5)$.
In dimension 1 one class is born at stage 6. Three more are born at stage 7, but two die at the same time. One more class from stage 7 dies at stage 9. So, we have $[6, \infty)$, $[7, 7)$, $[7, 7)$ and $[7, 9)$.

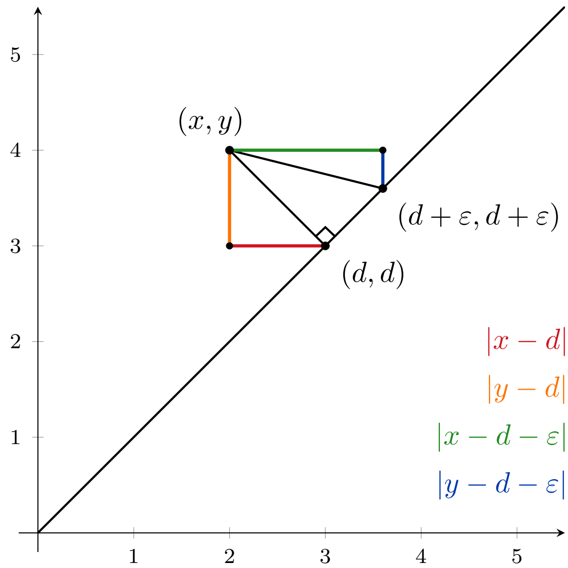
c.



d.



SOLUTION 48.



It is not difficult to guess that the point closest to (x, y) in the ∞-metric will be the same as in the Eulidean metric, ie. the point at the intersection of the diagonal and the line through (x, y), perpendicular to the diagonal. The coordinates of this point are

$$(d, d) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right),$$

so

$$|x-d| = \left|x - \frac{x+y}{2}\right| = \left|\frac{x-y}{2}\right| = \frac{|x-y|}{2}$$

and

$$|y-d| = \left|y - \frac{x+y}{2}\right| = \left|\frac{y-x}{2}\right| = \frac{|x-y|}{2}.$$

We see that

$$\|(x, y) - (d, d)\|_\infty = \max\left\{\frac{|x-y|}{2}, \frac{|x-y|}{2}\right\} = \frac{|x-y|}{2}.$$

For any other point $(d+\varepsilon, d+\varepsilon)$ we have

$$|x-d-\varepsilon| = \left|x - \frac{x+y}{2} - \varepsilon\right| = \left|\frac{x-y}{2} - \varepsilon\right| = \frac{|x-y-2\varepsilon|}{2}$$

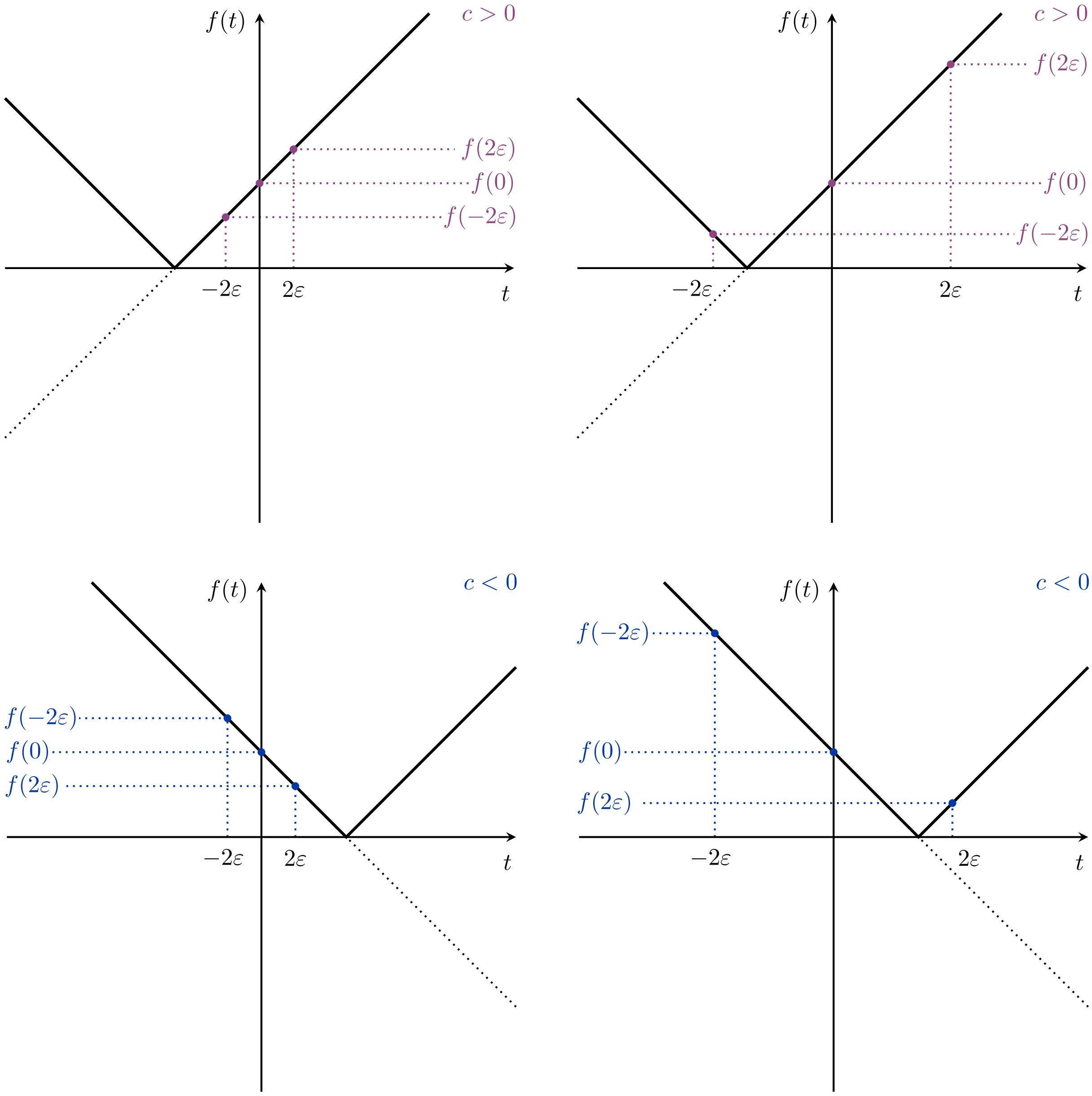
and

$$|y-d-\varepsilon| = \left|y - \frac{x+y}{2} - \varepsilon\right| = \left|\frac{y-x}{2} - \varepsilon\right| = \frac{|y-x-2\varepsilon|}{2} = \frac{|x-y+2\varepsilon|}{2},$$

so

$$\|(x,y)-(d+\varepsilon,d+\varepsilon)\|_\infty = \max\left\{\frac{|x-y-2\varepsilon|}{2}, \frac{|x-y+2\varepsilon|}{2}\right\}.$$

To see how $\|(x,y)-(d+\varepsilon,d+\varepsilon)\|_\infty$ compares to $\|(x,y)-(d,d)\|_\infty$, we need to compare $|x-y|$, $|x-y+2\varepsilon|$ and $|x-y-2\varepsilon|$. Recall that $x-y \in \mathbb{R}$ is a constant. Let $c = x-y$ and consider the function $f(t) = |c+t|$. We need to compare the values of $f(-2\varepsilon)$, $f(0)$ and $f(2\varepsilon)$. As the following figures show, at least one of $f(-2\varepsilon)$ and $f(2\varepsilon)$ is greater than $f(0)$, so at least one of $|x-y+2\varepsilon|$ and $|x-y-2\varepsilon|$ is greater than $|x-y|$.



In the figures we assume $\varepsilon > 0$. In this case

- if $c = x - y > 0$, then $f(2\varepsilon) > f(0)$,
- if $c = x - y < 0$, then $f(-2\varepsilon) > f(0)$.

If $\varepsilon < 0$, then

- if $c = x - y > 0$, then $f(-2\varepsilon) > f(0)$,
- if $c = x - y < 0$, then $f(2\varepsilon) > f(0)$.

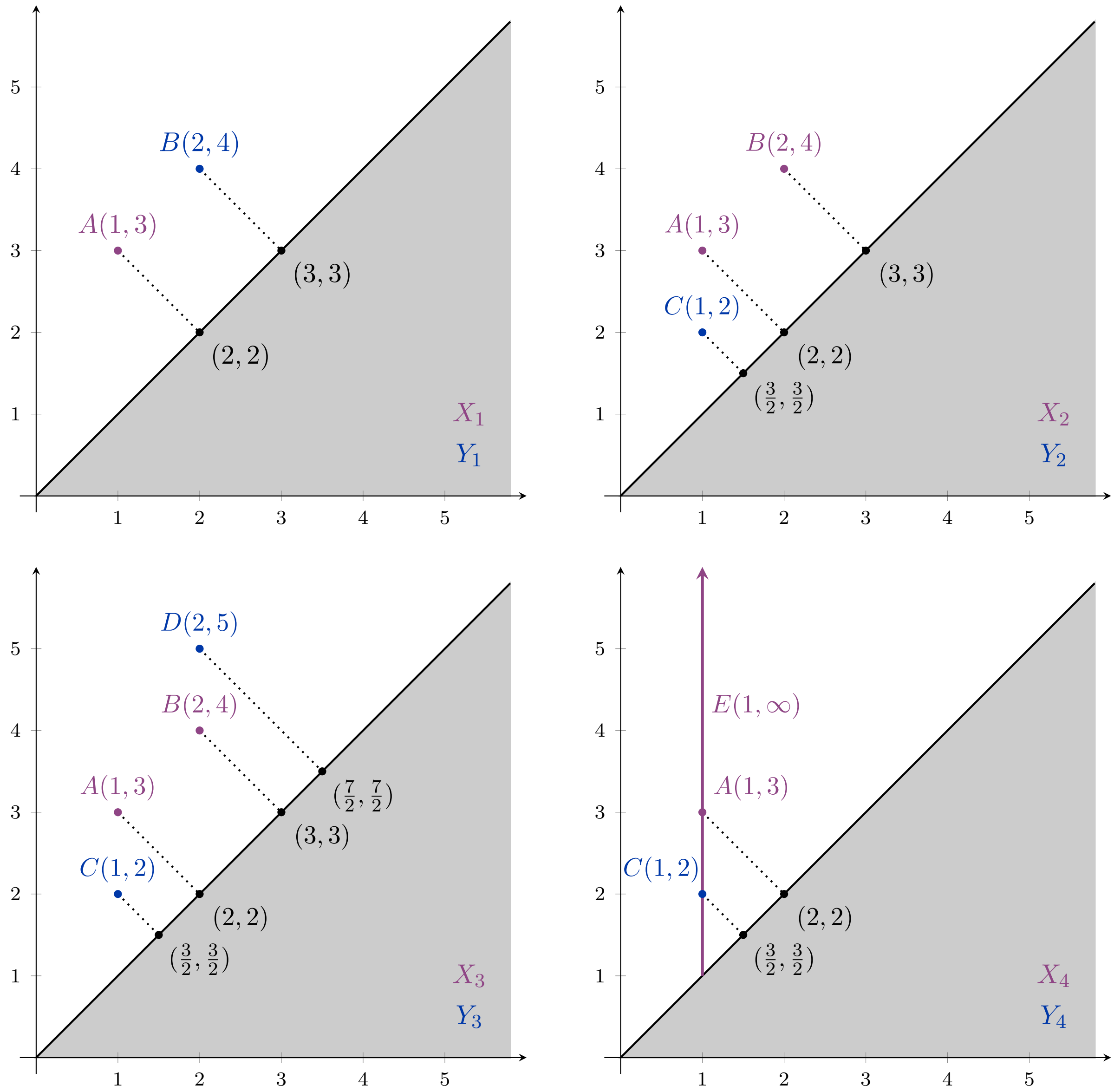
We conclude that

$$\|(x, y) - (d + \varepsilon, d + \varepsilon)\|_\infty > \|(x, y) - (d, d)\|_\infty,$$

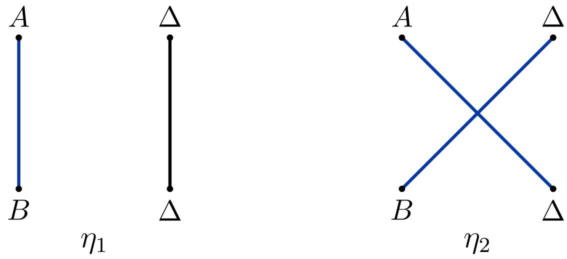
so we have shown that the closest point to (x, y) in L_∞ norm really is

$$(d, d) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right).$$

Solution 49.



a. There are two possible maps $\eta_i \colon X_1 \to Y_1$ as shown in the following figure.



We have

$$\begin{aligned}
\|A - \eta_1(A)\|_\infty &= \|A - B\|_\infty &&= \max\{|1-2|, |3-4|\} &&= 1,\\
\\
\|A - \eta_2(A)\|_\infty &= \|A - (2,2)\|_\infty &&= \max\{|1-2|, |3-2|\} &&= 1,\\
\|\Delta - \eta_2(\Delta)\|_\infty &= \|(3,3) - B\|_\infty &&= \max\{|2-3|, |4-3|\} &&= 1,
\end{aligned}$$

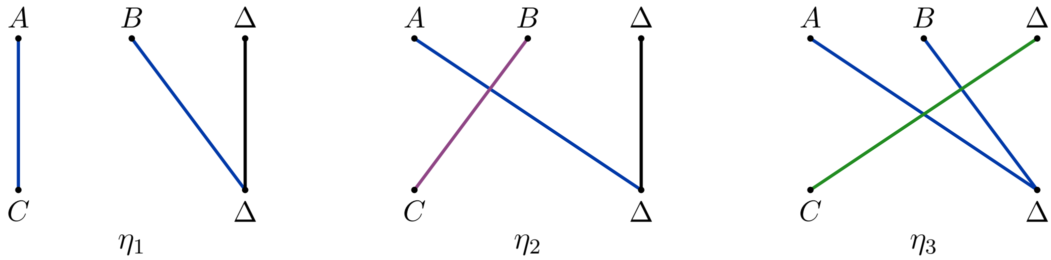
so

$$\sup_{x \in X_1} \|x - \eta_1(x)\|_\infty = 1 \quad \text{and} \quad \sup_{x \in X_1} \|x - \eta_2(x)\|_\infty = 1.$$

We get

$$W_\infty(X_1, Y_1) = \inf_{\eta\colon X_1 \to Y_1} \left(\sup_{x \in X_1} \|x - \eta(x)\|_\infty \right) = \inf\{1, 1\} = 1.$$

b. In this case there are three possible maps $\eta_i\colon X_2 \to Y_2$ as shown in the following figure.



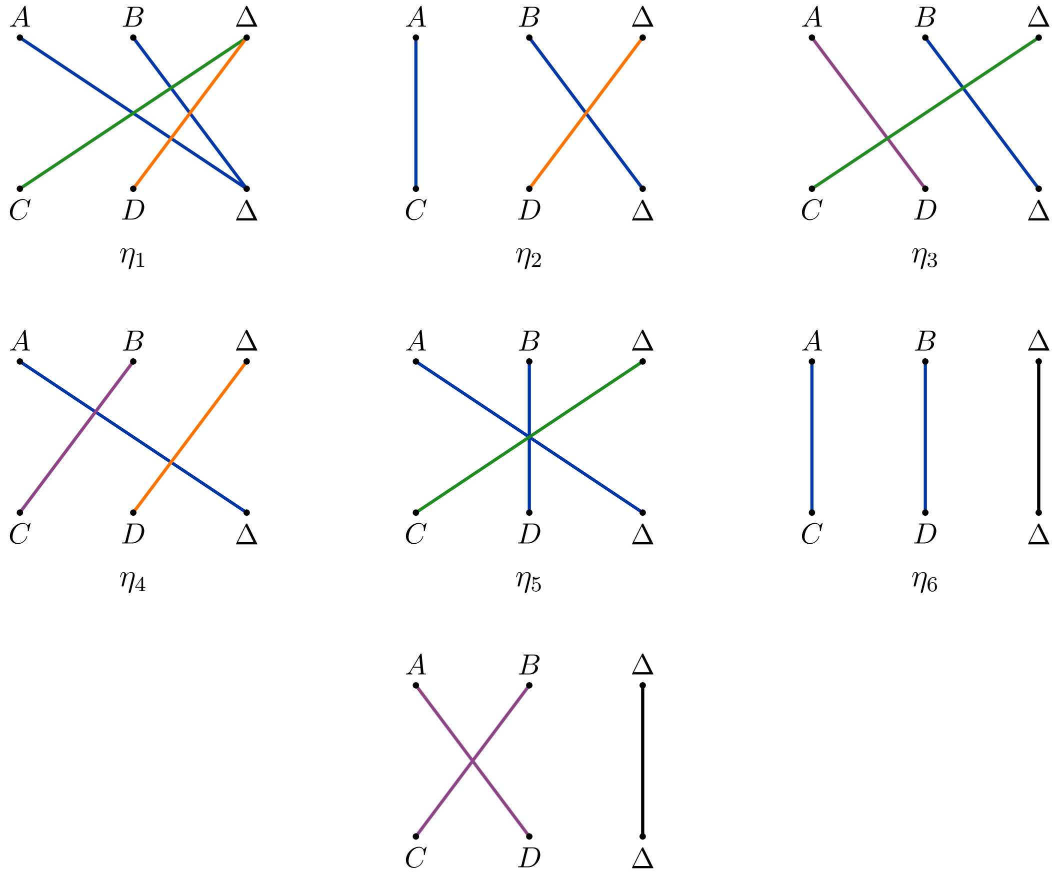
This time we have

$$\begin{aligned}
\|A - \eta_1(A)\|_\infty &= \|A - C\|_\infty &&= \max\{|1-1|, |3-2|\} &&= 1,\\
\|B - \eta_1(B)\|_\infty &= \|B - (3,3)\|_\infty &&= \max\{|2-3|, |4-3|\} &&= 1,\\
\\
\|A - \eta_2(A)\|_\infty &= \|A - (2,2)\|_\infty &&= \max\{|1-2|, |3-2|\} &&= 1,\\
\|B - \eta_2(B)\|_\infty &= \|B - C\|_\infty &&= \max\{|2-1|, |4-2|\} &&= 2,\\
\\
\|A - \eta_3(A)\|_\infty &= \|A - (2,2)\|_\infty &&= \max\{|1-2|, |3-2|\} &&= 1,\\
\|B - \eta_3(B)\|_\infty &= \|B - (3,3)\|_\infty &&= \max\{|2-3|, |4-3|\} &&= 1,\\
\|\Delta - \eta_3(\Delta)\|_\infty &= \|(\tfrac{3}{2}, \tfrac{3}{2}) - C\|_\infty &&= \max\{|\tfrac{3}{2}-1|, |\tfrac{3}{2}-2|\} &&= \tfrac{1}{2},
\end{aligned}$$

so $\sup_{x \in X_2} \|x - \eta_1(x)\|_\infty = 1$, $\sup_{x \in X_2} \|x - \eta_2(x)\|_\infty = 2$ and $\sup_{x \in X_2} \|x - \eta_3(x)\|_\infty = 1$. We get

$$W_\infty(X_2, Y_2) = \inf\{1, 2, 1\} = 1.$$

c. There are seven possible maps $\eta_i\colon X_3 \to Y_3$.



η_7

This time the distances are 2, $\frac{3}{2}$, 1 or $\frac{1}{2}$, so

$$W_\infty(X_3, Y_3) = \inf\{\tfrac{3}{2}, \tfrac{3}{2}, 2, 2, 1, 1, 2\} = 1.$$

d. We cannot compare X_4 and Y_4, because X_4 has one infinite class while Y_4 has none.

SOLUTION 50.

The maps η_i are the same as in the previous problem, we just need to recalculate the distances.

a. We have

$$\|A - \eta_1(A)\|_\infty^1 = \|A - B\|_\infty = \max\{|1-2|, |3-4|\} = 1,$$

$$\begin{aligned} \|A - \eta_2(A)\|_\infty^1 &= \|A - (2,2)\|_\infty = \max\{|1-2|, |3-2|\} = 1, \\ \|\Delta - \eta_2(\Delta)\|_\infty^1 &= \|(3,3) - B\|_\infty = \max\{|2-3|, |4-3|\} = 1, \end{aligned}$$

so

$$\sum_{x \in X_1} \|x - \eta_1(x)\|_\infty^1 = 1 \quad \text{and} \quad \sum_{x \in X_1} \|x - \eta_2(x)\|_\infty^1 = 1 + 1 = 2.$$

We get

$$W_1(X_1, Y_1) = \inf_{\eta\colon X_1 \to Y_1} \left(\sum_{x \in X_1} \|x - \eta(x)\|_\infty^1 \right) = \inf\{1, 2\} = 1.$$

b. For $q = 2$ we get

$$\begin{array}{rclclcl} \|A - \eta_1(A)\|_\infty^2 &=& \|A - B\|_\infty^2 &=& \max\{|1-2|, |3-4|\}^2 &=& 1, \\ \\ \|A - \eta_2(A)\|_\infty^2 &=& \|A - (2,2)\|_\infty^2 &=& \max\{|1-2|, |3-2|\}^2 &=& 1, \\ \|\Delta - \eta_2(\Delta)\|_\infty^2 &=& \|(3,3) - B\|_\infty^2 &=& \max\{|2-3|, |4-3|\}^2 &=& 1, \end{array}$$

so

$$\sum_{x \in X_1} \|x - \eta_1(x)\|_\infty^2 = 1 \quad \text{and} \quad \sum_{x \in X_1} \|x - \eta_2(x)\|_\infty^2 = 1 + 1 = 2.$$

We get

$$W_2(X_1, Y_1) = \left(\inf_{\eta\colon X_1 \to Y_1} \sum_{x \in X_1} \|x - \eta(x)\|_\infty^2 \right)^{\frac{1}{2}} = \sqrt{\inf\{1,2\}} = \sqrt{1} = 1.$$

c. This time we have

$$\begin{array}{rclclcl} \|A - \eta_1(A)\|_\infty^1 &=& \|A - C\|_\infty &=& \max\{|1-1|, |3-2|\} &=& 1, \\ \|B - \eta_1(B)\|_\infty^1 &=& \|B - (3,3)\|_\infty &=& \max\{|2-3|, |4-3|\} &=& 1, \\ \\ \|A - \eta_2(A)\|_\infty^1 &=& \|A - (2,2)\|_\infty &=& \max\{|1-2|, |3-2|\} &=& 1, \\ \|B - \eta_2(B)\|_\infty^1 &=& \|B - C\|_\infty &=& \max\{|2-1|, |4-2|\} &=& 2, \\ \\ \|A - \eta_3(A)\|_\infty^1 &=& \|A - (2,2)\|_\infty &=& \max\{|1-2|, |3-2|\} &=& 1, \\ \|B - \eta_3(B)\|_\infty^1 &=& \|B - (3,3)\|_\infty &=& \max\{|2-3|, |4-3|\} &=& 1, \\ \|\Delta - \eta_3(\Delta)\|_\infty^1 &=& \|(\frac{3}{2}, \frac{3}{2}) - C\|_\infty &=& \max\{|\frac{3}{2}-1|, |\frac{3}{2}-2|\} &=& \frac{1}{2}, \end{array}$$

so

$$\sum_{x \in X_2} \|x - \eta_1(x)\|_\infty^1 = 1 + 1 = 2,$$

$$\sum_{x \in X_2} \|x - \eta_2(x)\|_\infty^1 = 1 + 2 = 3$$

and

$$\sum_{x \in X_2} \|x - \eta_3(x)\|_\infty^1 = 1 + 1 + \tfrac{1}{2} = \tfrac{5}{2}.$$

We get

$$W_1(X_2, Y_2) = \inf\{2, 3, \tfrac{5}{2}\} = 2.$$

d. When $q = 2$ we have

$$\begin{aligned}
\|A - \eta_1(A)\|_\infty^2 &= \|A - C\|_\infty^2 &= \max\{|1-1|, |3-2|\}^2 &= 1,\\
\|B - \eta_1(B)\|_\infty^2 &= \|B - (3,3)\|_\infty^2 &= \max\{|2-3|, |4-3|\}^2 &= 1,\\
\\
\|A - \eta_2(A)\|_\infty^2 &= \|A - (2,2)\|_\infty^2 &= \max\{|1-2|, |3-2|\}^2 &= 1,\\
\|B - \eta_2(B)\|_\infty^2 &= \|B - C\|_\infty^2 &= \max\{|2-1|, |4-2|\}^2 &= 2^2 = 4,\\
\\
\|A - \eta_3(A)\|_\infty^2 &= \|A - (2,2)\|_\infty^2 &= \max\{|1-2|, |3-2|\}^2 &= 1,\\
\|B - \eta_3(B)\|_\infty^2 &= \|B - (3,3)\|_\infty^2 &= \max\{|2-3|, |4-3|\}^2 &= 1,\\
\|\Delta - \eta_3(\Delta)\|_\infty^2 &= \|(\tfrac{3}{2}, \tfrac{3}{2}) - C\|_\infty^2 &= \max\{|\tfrac{3}{2}-1|, |\tfrac{3}{2}-2|\}^2 &= \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4},
\end{aligned}$$

so

$$\sum_{x \in X_2} \|x - \eta_1(x)\|_\infty^2 = 1 + 1 = 2,$$

$$\sum_{x \in X_2} \|x - \eta_2(x)\|_\infty^2 = 1 + 4 = 5$$

and

$$\sum_{x \in X_2} \|x - \eta_3(x)\|_\infty^2 = 1 + 1 + \tfrac{1}{4} = \tfrac{9}{4}.$$

We get

$$W_2(X_2, Y_2) = \sqrt{\inf\{2, 5, \tfrac{9}{4}\}} = \sqrt{2}.$$

e. For $q = 1$ we have

$$\begin{aligned}
\sum_{x \in X_3} \|x - \eta_1(x)\|_\infty^1 &= 1 + 1 + \tfrac{1}{2} + \tfrac{3}{2} &= 4,\\
\sum_{x \in X_3} \|x - \eta_2(x)\|_\infty^1 &= &= 1 + 1 + \tfrac{3}{2} = \tfrac{7}{2},\\
\sum_{x \in X_3} \|x - \eta_3(x)\|_\infty^1 &= 1 + \tfrac{1}{2} + 2 &= \tfrac{7}{2},\\
\sum_{x \in X_3} \|x - \eta_4(x)\|_\infty^1 &= 1 + \tfrac{3}{2} + 2 &= \tfrac{9}{2},\\
\sum_{x \in X_3} \|x - \eta_5(x)\|_\infty^1 &= 1 + 1 + \tfrac{1}{2} &= \tfrac{5}{2},\\
\sum_{x \in X_3} \|x - \eta_6(x)\|_\infty^1 &= 1 + 1 &= 2,\\
\sum_{x \in X_3} \|x - \eta_7(x)\|_\infty^1 &= 2 + 2 &= 4,
\end{aligned}$$

so

$$W_1(X_3, Y_3) = \inf\{4, \tfrac{7}{2}, \tfrac{7}{2}, \tfrac{9}{2}, \tfrac{5}{2}, 2, 4\} = 2.$$

f. For $q = 2$ we have

$$\begin{aligned}
\sum_{x\in X_3} \|x-\eta_1(x)\|_\infty^2 &= 1+1+\tfrac{1}{4}+\tfrac{9}{4} &&= \tfrac{9}{2},\\
\sum_{x\in X_3} \|x-\eta_2(x)\|_\infty^2 &= \quad = 1+1+\tfrac{9}{4} = \tfrac{17}{4},\\
\sum_{x\in X_3} \|x-\eta_3(x)\|_\infty^2 &= 1+\tfrac{1}{4}+4 &&= \tfrac{21}{4},\\
\sum_{x\in X_3} \|x-\eta_4(x)\|_\infty^2 &= 1+\tfrac{9}{4}+4 &&= \tfrac{29}{4},\\
\sum_{x\in X_3} \|x-\eta_5(x)\|_\infty^2 &= 1+1+\tfrac{1}{4} &&= \tfrac{9}{4},\\
\sum_{x\in X_3} \|x-\eta_6(x)\|_\infty^2 &= 1+1 &&= 2,\\
\sum_{x\in X_3} \|x-\eta_7(x)\|_\infty^2 &= 4+4 &&= 8,
\end{aligned}$$

so

$$W_2(X_3, Y_3) = \sqrt{\inf\{\tfrac{9}{2}, \tfrac{17}{4}, \tfrac{21}{4}, \tfrac{29}{4}, \tfrac{9}{4}, 2, 8\}} = \sqrt{2}.$$

With only a few lines of code we can quickly check our results. For example, running

```
import numpy as np
import gudhi
import gudhi.wasserstein

diagX3 = np.array([[1,3],[2,4]])
diagY3 = np.array([[1,2],[2,5]])

bottle = gudhi.bottleneck_distance(diagX3, diagY3)
wasser1 = gudhi.wasserstein.wasserstein_distance(diagX3, diagY3, order=1.)
wasser2 = gudhi.wasserstein.wasserstein_distance(diagX3, diagY3, order=2.)

print("Bottleneck distance between X3 and Y3 is:  "+str(bottle))
print("Wasserstein distance between X3 and Y3 for q=1 is:  "+str(wasser1))
print("Wasserstein distance between X3 and Y3 for q=2 is:  "+str(wasser2))
```

in Python 3.5.8 tells us that

```
Bottleneck distance between X3 and Y3 is:  1.0
Wasserstein distance between X3 and Y3 for q=1 is:  2.0
Wasserstein distance between X3 and Y3 for q=2 is:  1.4142135623730951
```

Solution 51.

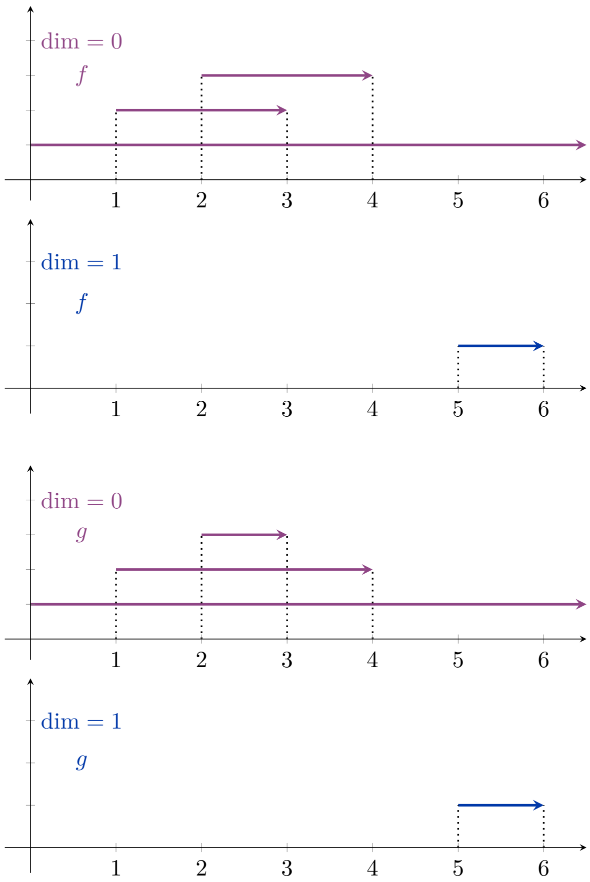
a. For f the correct sequence is

$$B,\ A,\ C,\ AB,\ AC,\ BC,\ ABC$$

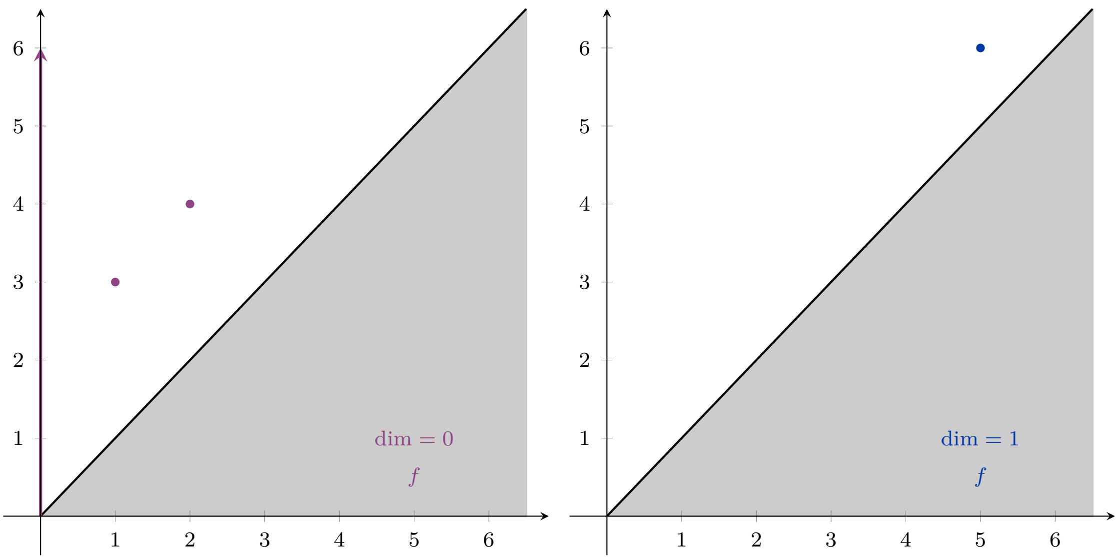
and for g we get

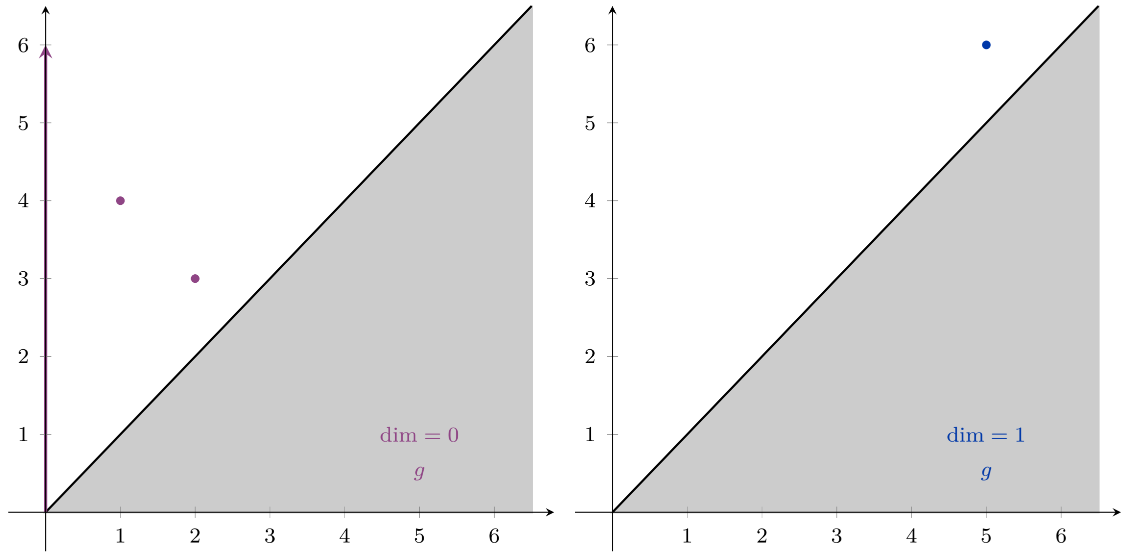
$$A,\ B,\ C,\ BC,\ AC,\ AB,\ ABC.$$

b. For f we have 0-dimensional classes $[0, \infty)$, $[1, 3)$ and $[2, 4)$ and the 1-dimensional class $[5, 6)$. For g the 0-dimensional classes change to $[0, \infty)$, $[1, 4)$ and $[2, 3)$ and the 1-dimensional class is still $[5, 6)$. The barcodes are shown in the following figure.



c. The persistence diagrams are shown below.





d. The persistence diagrams in dimension 1 are exactly the same, so

$$W_\infty(\mathrm{Dgm}_1(f), \mathrm{Dgm}_1(g)) = 0.$$

In dimension 0 we should match the finite classes in pairs (in either of the two possible ways) to get the smallest possible distances $\|x - \eta(x)\|_\infty = 1$. We can conclude that

$$W_\infty(\mathrm{Dgm}_0(f), \mathrm{Dgm}_0(g)) = 1.$$

If we wanted to check all possible matchings, they would look a lot like the matchings between X_3 and Y_3 in Problem 49, only some of the distances would be slightly different.

e. The two boundary matrices are

$$D_f = \begin{array}{cc} & \begin{array}{ccccccc} B & A & C & AB & AC & BC & ABC \end{array} \\ \begin{array}{c} B \\ A \\ C \\ AB \\ AC \\ BC \\ ABC \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

and

$$D_g = \begin{array}{cc} & \begin{array}{ccccccc} A & B & C & BC & AC & AB & ABC \end{array} \\ \begin{array}{c} A \\ B \\ C \\ BC \\ AC \\ AB \\ ABC \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

f. We have the following values for $\mathrm{low}(j)$ for D_f:

j	1	2	3	4	5	6	7
$\mathrm{low}(j)$	–	–	–	2	3	3	6

Following the algorithm to reduce D_f we get

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{\mathbf{1}} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{\mathbf{1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{\mathbf{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In order to correct the value of low(6) we have first added the fifth and then the fourth column to the sixth (working in $\mathbb{Z}_2$). Once this is done, we have the values

j	1	2	3	4	5	6	7
low(j)	–	–	–	2	3	–	6

and the matrix is reduced to R_f.

For D_g the values of low(j) are:

j	1	2	3	4	5	6	7
low(j)	–	–	–	3	3	2	6

Following the algorithm to reduce D_g we get

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \underline{\mathbf{1}} & 0 & 0 \\ 0 & 0 & 0 & \underline{\mathbf{1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{\mathbf{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We had to add the fourth column to the fifth and then the fifth to the sixth. After this the values are

j	1	2	3	4	5	6	7
low(j)	–	–	–	3	2	–	6

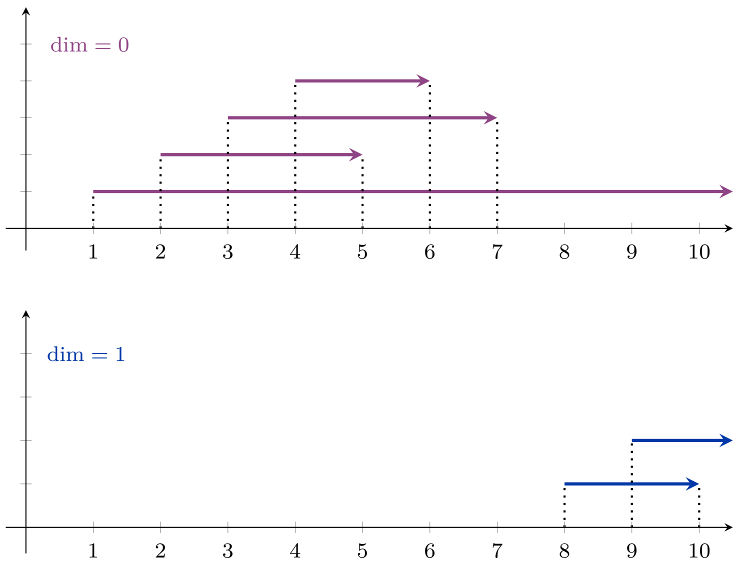
and the matrix is reduced to R_g. If we index the columns from 0 to 6 as per function values, we see that the lowest ones are in positions (1, 3), (2, 4), (5, 6) for R_f and (1, 4), (2, 3), (5, 6) for R_g, which is consistent with the persistence classes computed above. For the infinite classes we need to consider all zero columns with index i such that the row with index i does not contain a lowest one. For R_f and $i \in \{0, 1, 2, 5\}$ this happens when $i = 0$, which means we have an infinite class $[0, \infty)$, and the same is true for R_g.

SOLUTION 52.

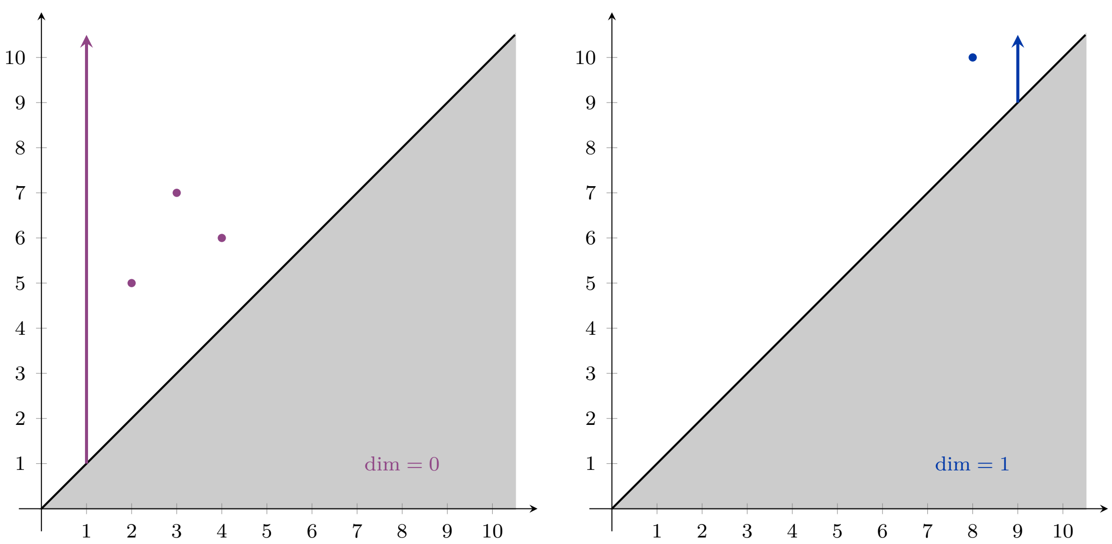
a. The correct order for the simplices is

$$A,\ B,\ C,\ D,\ AB,\ BD,\ BC,\ CD,\ AD,\ BCD.$$

b. In dimension 0 we get the classes $[1,\ \infty)$, $[2,\ 5)$, $[3,\ 7)$ and $[4,\ 6)$. In dimension 1 the classes are $[8,\ 10)$ and $[9,\ \infty)$.



c. The corresponding persistence diagrams are shown in the following figure.



d. The boundary matrix is

$$D_f = \begin{array}{cc} & \begin{array}{cccccccccc} A & B & C & D & AB & BD & BC & CD & AD & BCD \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ AB \\ BD \\ BC \\ CD \\ AD \\ BCD \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

e. The values of low(j) for D_f are:

j	1	2	3	4	5	6	7	8	9	10
low(j)	–	–	–	–	2	4	3	4	4	8

We need to add columns 6 to 8, 7 to 8, 6 to 9 and 5 to 9. When we finish we get

j	1	2	3	4	5	6	7	8	9	10
low(j)	–	–	–	–	2	4	3	–	–	8

and the reduced matrix is

$$R_f = \begin{array}{cc} & \begin{array}{cccccccccc} A & B & C & D & AB & BD & BC & CD & AD & BCD \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ AB \\ BD \\ BC \\ CD \\ AD \\ BCD \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

We see that the lowest ones are (2, 5), (4, 6), (3, 7) and (8, 10), just as they should be. For $i \in \{1, 2, 3, 4, 8, 9\}$ the columns contain only zeros, and out of these the rows with indices 1 and 9 do not contain the lowest ones, which gives us the infinite classes $[1, \infty)$ and $[9, \infty)$.

SOLUTION 53.

Using the order of the simplices given in problem 46 (ie. the order in which they are added in the filtration) the boundary matrix is

$$D = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

We have the following values for $\text{low}(j)$:

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\text{low}(j)$	–	–	–	–	–	2	3	4	5	3	4	5	4	10	11	13	13

We add the seventh column to the tenth and then $\text{low}(10) = 2$, so we also add the sixth column to the tenth which causes column ten to become a zero-column. Next, we add the eighth column to the eleventh column to once again get $\text{low}(11) = 2$. Adding the sixth column everything in the eleventh cancels out. The same happens when we add first the ninth and then the sixth column to the twelfth column. When we add the eighth column to the thirteenth column, we get $\text{low}(13) = 3$, so this time we need to add the seventh column to cancel everything. Finally, we only need to add the sixteenth column to the seventeenth to get $\text{low}(17) = 11$. Adding the fifteenth column leaves us with $\text{low}(17) = 10$. We add the fourteenth column and finally get another zero-column. We now have the following values for $\text{low}(j)$:

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\text{low}(j)$	–	–	–	–	–	2	3	4	5	–	–	–	–	10	11	13	–

The reduced matrix is equal to

$$R = \begin{bmatrix}
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\underline{\mathbf{1}}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\underline{\mathbf{1}}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\underline{\mathbf{1}}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\underline{\mathbf{1}}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\underline{\mathbf{1}}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\underline{\mathbf{1}}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\underline{\mathbf{1}}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}.$$

Notice that we have the lowest ones for (i, j) equal to $(2, 6)$, $(3, 7)$, $(4, 8)$, $(5, 9)$, $(10, 14)$, $(11, 15)$ and $(13, 16)$. These are exactly the finite persistence classes we have identified in Problem 46. For the infinite classes we get $i \in \{1, 2, 3, 4, 5, 10, 11, 12, 13, 17\}$ for the zero columns. A quick look at the finite classes tells us that out of these $i \in \{2, 3, 4, 5, 10, 11, 13\}$ correspond to a lowest one, which means that

$$i \in \{1, 2, 3, 4, 5, 10, 11, 12, 13, 17\} \setminus \{2, 3, 4, 5, 10, 11, 13\} = \{1, 12, 17\}$$

correspond to infinite classes $[1, \infty)$, $[12, \infty)$ and $[17, \infty)$.

7. Morse theory

Solution 54.

a. For every simplex in K we list all faces with higher values and all cofaces with lower values in the following arrays. We see that there is at most one exception for every simplex, so F and G are Morse functions on K.

σ	A	B	C	D	E	AB	AD	BC	BD	CD	ABD
$\tau > \sigma$ with $F(\tau) \leq F(\sigma)$	–	AB	BC	BD	–	–	ABD	–	–	–	–
$\tau > \sigma$ with $F(\tau) \geq F(\sigma)$	–	–	–	–	–	B	–	C	D	–	AD

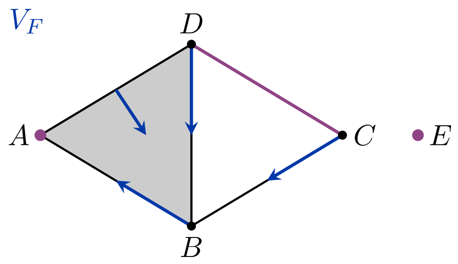
σ	A	B	C	D	E	AB	AD	BC	BD	CD	ABD
$\tau > \sigma$ with $G(\tau) \leq G(\sigma)$	–	BC	–	CD	–	ABD	–	–	–	–	–
$\tau > \sigma$ with $G(\tau) \geq G(\sigma)$	–	–	–	–	–	–	–	B	–	D	AB

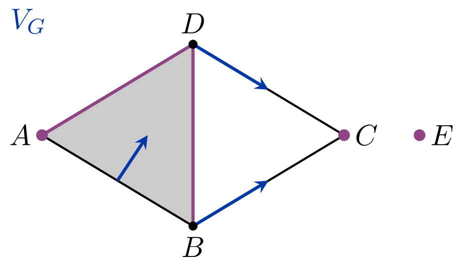
b. The critical simplices are those with no exceptions. For F these are A, E and CD. For G the critical simplices are A, C, E, AD and BD. The corresponding vector fields are

$$V_F = \{(B, AB), (C, BC), (D, BD), (AD, ABD)\}$$

and

$$V_G = \{(B, BC), (D, CD), (AB, ABD)\}.$$





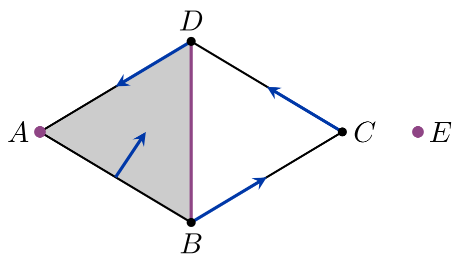
c. The maximal non-trivial gradient paths in V_F are

$$\begin{aligned}\alpha &: \quad C,\ BC,\ B,\ AB,\ A,\\ \beta &: \quad D,\ BD,\ B,\ AB,\ A,\\ \gamma &: \quad AD,\ ABD.\end{aligned}$$

The only two critical simplices we could potentially cancel are CD and A, but there are two paths from $\partial(CD)$ to A, namely α and β, so the cancellation is not possible. In V_G the maximal non-trivial gradient paths are

$$\begin{aligned}\alpha &: \quad D,\ CD,\ C,\\ \beta &: \quad B,\ BC,\ C,\\ \gamma &: \quad AB,\ ABD.\end{aligned}$$

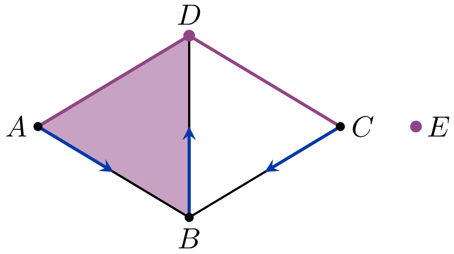
There are two paths, α and β, from $\partial(BD)$ to C, so we are not allowed to cancel BD and C. There are no paths from $\partial(BD)$ to A, so we cannot cancel BD and A. On the other hand there is only one path, α, from $\partial(AD)$ to C, so we can cancel the critical simplices AD and C by reversing the path α, pairing C to CD and D to AD. The resulting vector field is shown in the following figure.



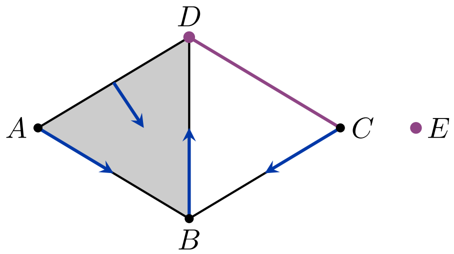
This vector field has only one critical simplex in dimension 0 for each connected component, which is as low as we can go. There is also a critical simplex in dimension 1 in one of the components, which cannot be cancelled, because we know the only critical simplex in dimension 0 must remain (and, also, this component is not contractible, so there has to be at least one more critical simplex other than the one in dimension 0). So, this vector field has the smallest possible number of critical simplices.
Note that there is also only one path from $\partial(AD)$ to A (the constant path c_A), so we could instead pair D to AD to obtain another vector field with a path A, AD, D, CD.

d. We know we must have one critical simplex in dimension 0 for each connected component, so E must be a critical simplex. There is only one simplex in dimension 2, so ABD is also critical. We can choose any of the remaining four vertices to be the second critical simplex in dimension 0. Let us choose D. Finally, we have to choose two critical simplices in dimension 1, for example AD and CD

(there are many other possibilities, of course). There is only one way to pair up the remaining non-critical simplices without creating more critical simplices. We obtain the vector field shown in the following figure.



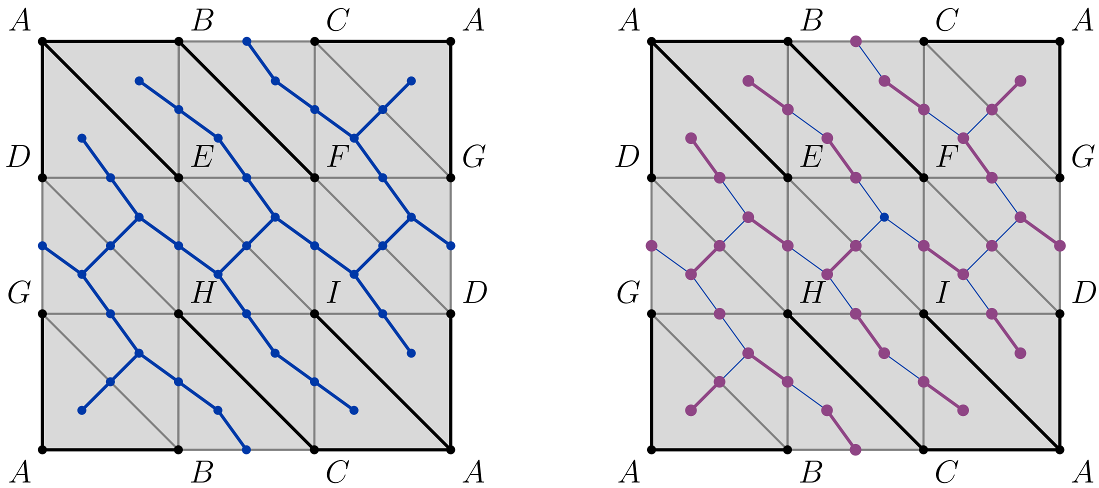
We can pair AD to ABD to get a vector field with $c_2 = 0$, $c_1 = 1$ and $c_0 = 2$.



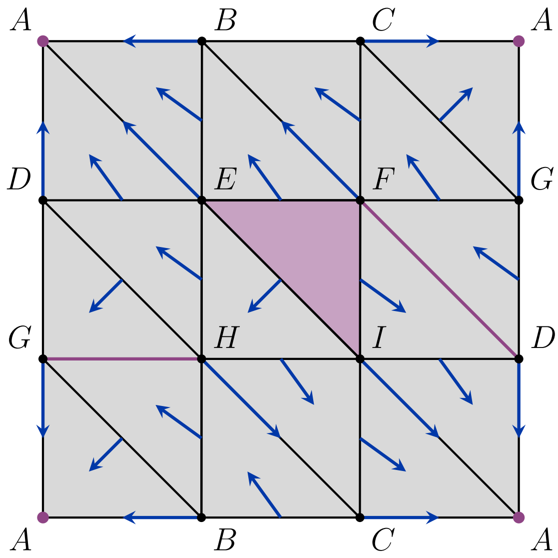
Solution 55.

a. The easiest way to get a 1-cycle that is a boundary is to compute the boundary of a 2-simplex. For example, $\partial(ABE) = AB + BE - AE$. A 1-cycle that is not a boundary is any representative of a non-trivial loop in K. For example, $AD + DG - AG$ is one of the two generators of $H_1(K)$, so it is not a boundary. There is also a 2-cycle that is not a boundary, namely the sum of all 2-simplices that generates $H_2(K; \mathbb{Z}_2) = \mathbb{Z}_2$.

b. We will try to construct a discrete gradient vector field on K with the optimal number of critical simplices. We begin by choosing a spanning tree of the 1-skeleton of K. In our example the base of the tree is at A. The spanning tree contains all 9 vertices and 8 of the edges. The 1-paths in our gradient field will run along this spanning tree towards A. Next, we connect the centers of the 2-simplices with the centers of those of their edges that do not belong in the spanning tree. The resulting graph is shown in blue on the left in the figure below. Finally, we choose a maximal matching in this graph. This matching (shown in purple on the right) will give us the pairs that will make up the 2-paths in our vector field.



From this information we get the vector field shown below.



c. Our vector field has critical simplices A, DF, GH and EFI, so $c_0 = 1$, $c_1 = 2$, $c_2 = 1$ and

$$\chi(K) = c_0 - c_1 + c_2 = 1 - 2 + 1 = 0.$$

d. The groups in the Morse chain complex are

$$M_2 = \langle EFI \rangle, \quad M_1 = \langle DF, GH \rangle \quad \text{and} \quad M_0 = \langle A \rangle.$$

To compute $\partial_2 \colon M_2 \to M_1$, we need to count the 2-paths from $\partial(EFI) = EF + FI + EI$ to DF and GH. There is one 2-path from FI to DF and one 2-path from EI to DF, so that is two 2-paths to DF. There are also two 2-paths from EI to GH, so

$$\partial_2(EFI) = (1+1) \cdot DF + 2 \cdot GH = 0 \cdot DF + 0 \cdot GH = 0.$$

For $\partial_1 \colon M_1 \to M_0$ we need to count the 1-paths from $\partial(DF) = D + F$ and $\partial(GH) = G + H$ to A. For each of D, F, G and H, there is exactly one 1-path to A (the shortest path in the spanning tree we used to construct V). So,

$$\begin{aligned} \partial_1(DF) &= (1+1) \cdot A = 0, \\ \partial_1(GH) &= (1+1) \cdot A = 0. \end{aligned}$$

We can now compute

$$\begin{aligned} H_2(M; \mathbb{Z}_2) &= \frac{Z_2(M)}{B_2(M)} = \frac{\ker \partial_2}{\mathrm{im} \partial_3} = \ker \partial_2 = \langle EFI \rangle \cong \mathbb{Z}_2, \\ H_1(M; \mathbb{Z}_2) &= \frac{Z_1(M)}{B_1(M)} = \frac{\ker \partial_1}{\mathrm{im} \partial_2} = \ker \partial_1 = \langle DF, GH \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \\ H_0(M; \mathbb{Z}_2) &= \frac{Z_0(M)}{B_0(M)} = \frac{\ker \partial_0}{\mathrm{im} \partial_1} = \ker \partial_0 = \langle A \rangle \cong \mathbb{Z}_2. \end{aligned}$$

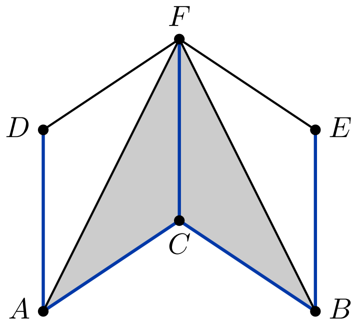
We conclude that the Betti numbers are $b_2 = 1$, $b_1 = 2$ and $b_0 = 1$. We have managed to find a vector field with the smallest possible number of critical simplices, so $c_i = b_i$. For a different vector field with more critical simplices we could only expect to have $c_i \geq b_i$ for $i = 0, 1, 2$ (by the Weak Morse Inequalities).

Solution 56.

a. We need to assign values to the edges and faces of X.

A	B	C	D	E	F	AC	AD	AF	BC	BE	BF	CF	DF	EF	ACF	BCF
0	0	2	4	4	6	–	–	–	–	–	–	–	–	–	–	–

First, we can extend the values to part of the 1-skeleton of X by choosing a spanning tree and assigning to most of the edges the average of the values at the two endpoints. Note that we cannot have $F(AC) = F(BC) = 1$ because that would mean two exceptions at C. Instead, we make the edge BC critical by defining $F(BC) = \max\{F(B), F(C)\} + 1 = 3$. If we were free to pick the values at the vertices as well, we could assign the averages to all the edges in the spanning tree, but in our case, the values at the vertices are predetermined, so we should really use the algorithm for extending the function one simplex or one pair of simplices at a time, as we will do for the rest of the complex.



A	B	C	D	E	F	AC	AD	AF	BC	BE	BF	CF	DF	EF	ACF	BCF
0	0	2	4	4	6	1	2	–	3	2	–	4	–	–	–	–

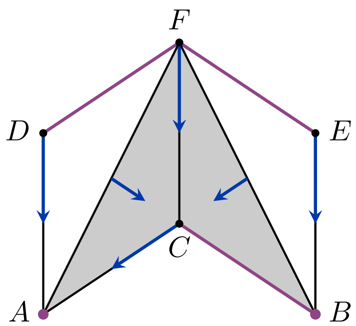
Next, we can add the the values to the two pairs of simplices (AF, ACF) and (BF, BCF) where the 1-simplex is a free face of the 2-simplex. The maximum value of the function before these additions is 6, so we can define $F(AF) = F(BF) = 8$ and $F(ACF) = F(BCF) = 7$. With these values none of the added simplices are critical.

A	B	C	D	E	F	AC	AD	AF	BC	BE	BF	CF	DF	EF	ACF	BCF
0	0	2	4	4	6	1	2	8	3	2	8	4	–	–	7	7

Finally, we have to assign the values to the edges DF and EF. Adding each of these edges changes the homology of the simplicial complex, so they will both become critical edges of F. The maximum of F so far is 8, so let us define $F(DF) = F(EF) = 9$. One possible Morse function F is

A	B	C	D	E	F	AC	AD	AF	BC	BE	BF	CF	DF	EF	ACF	BCF
0	0	2	4	4	6	1	2	8	1	2	8	4	9	9	7	7

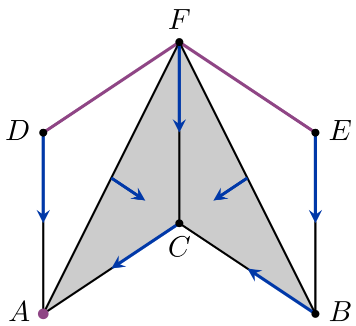
b. The corresponding vector field V_F is shown in the following figure.



This vector field has three critical simplices in dimension 1 and 2 critical simplices in dimension 0. It also has 6 vertices, 9 edges and 2 triangles. The Euler characteristic of this complex is

$$\chi(X) = 6 - 9 + 2 = 2 - 3 = -1.$$

c. There is only one path from the boundary $\partial(FE)$ of the critical edge FE to the critical vertex B, so we could cancel FE and B by pairing B with BE and E with EF. Another option would be to cancel BC and B (the only path between them is the constant path) by pairing B with BC. Let us choose the latter. The resulting vector field is shown in the following figure.



This vector field has two critical edges and one critical vertex. We know that there has to be at least one critical vertex, so no other cancellations are possible. The Euler characteristic is still $\chi(X) = 1 - 2 = -1$.

Solution 57.

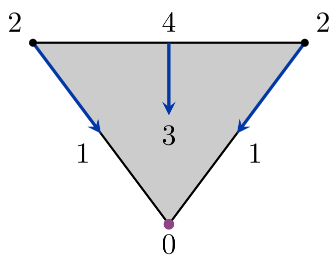
a. Let F be a discrete Morse function on X and assume that $F(\sigma)$ is minimal for some simplex σ with $n = \dim(\sigma) \geq 1$. Consider the $n+1$ faces $\tau_i < \sigma$ with $\dim(\tau_i) = \dim(\sigma) - 1$ (that is, all $n+1$ faces of codimension 1). Since $F(\sigma)$ is the minimum, we have $F(\tau) \geq F(\sigma)$, so

$$|\{\tau^{(n-1)} < \sigma^{(n)} \mid F(\tau) \geq F(\sigma)\}| = n + 1 \geq 1 + 1 = 2.$$

This contradicts the assumption that F is Morse, so $n < 1$. In other words, $n = 0$ and σ can only be a vertex.

b. A triangulated surface M without boundary has a dual decomposition M^*, where every d-dimensional simplex is replaced by a $(2-d)$-dimensional simplex. If F is a Morse function on M, then $-F$ is a Morse function on the dual complex M^*. The maximum of F is the minimum of $-F$, and that minimum occurs at a vertex of M^* which corresponds to a 2-simplex of M. So, the maximum of F occurs at a 2-simplex.

c. If M is not a manifold without boundary, then the conclusion from the previous point is no longer true. An example of such a simplicial complex X is shown below. This complex is still a manifold but it has a non-empty boundary and the maximum occurs at an edge.



SOLUTION 58.

The chain groups are generated by the critical simplices in each dimension, so $M_1 = \langle AH, EF\rangle$ and $M_0 = \langle E\rangle$. The boundary homomorphisms connect the chain groups into a sequence

$$0 \xrightarrow{D_2} M_1 \xrightarrow{D_1} M_0 \xrightarrow{D_0} 0.$$

We have

$$D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D_1 = \begin{bmatrix} \langle \partial(AH), E\rangle & \langle \partial(EF), E\rangle \end{bmatrix}, D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

To calculate $\langle \partial(AH), E\rangle$ we need to find all paths from $\sigma \in \{A, H\}$ to E. For $\sigma = A$ we have just one path

$$\alpha\colon A,\ AB,\ B,\ BC,\ C,\ CD,\ D,\ DE,\ E,$$

with multiplicity $m(\alpha) = 1$. For $\sigma = H$ we also have a unique path

$$\beta\colon H,\ GH,\ G,\ FG,\ F,\ DF,\ D,\ DE,\ E,$$

with multiplicity $m(\beta) = 1$. So,

$$\begin{aligned} \langle \partial(AH), E\rangle &= \langle \partial(AH), A\rangle m(\alpha) + \langle \partial(AH), H\rangle m(\beta) = \\ &= \langle H - A, A\rangle m(\alpha) + \langle H - A, H\rangle m(\beta) = \\ &= -1 \cdot 1 + 1 \cdot 1 = 0. \end{aligned}$$

To compute $\langle \partial(EF), E\rangle$ we need to find all paths from $\sigma \in \{E, F\}$ to E. For $\sigma = E$ we have only the constant path γ with multiplicity $m(\gamma) = 1$. For $\sigma = F$ we have a unique path

$$\delta\colon F,\ DF,\ D,\ DE,\ E,$$

with multiplicity $m(\delta) = 1$. So,

$$\begin{aligned} \langle \partial(EF), E\rangle &= \langle \partial(EF), E\rangle m(\gamma) + \langle \partial(EF), F\rangle m(\delta) = \\ &= \langle F - E, E\rangle m(\gamma) + \langle F - E, F\rangle m(\delta) = \\ &= -1 \cdot 1 + 1 \cdot 1 = 0. \end{aligned}$$

We have calculated the coefficients of D_1, so we now know that the boundary matrices are

$$D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

We can now compute

$$\begin{aligned} H_1(\mathbb{R}P^2; \mathbb{Z}) &= \frac{\ker D_1}{\operatorname{im} D_2} = \frac{\langle AH, EF\rangle}{\langle 0\rangle} = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2, \\ H_0(\mathbb{R}P^2; \mathbb{Z}) &= \frac{\ker D_0}{\operatorname{im} D_1} = \frac{\langle E\rangle}{\langle 0\rangle} = \mathbb{Z}. \end{aligned}$$

SOLUTION 59.

The chain groups are generated by the critical simplices in each dimension, so $M_2 = \langle ABC\rangle$, $M_1 = \langle BF, CE\rangle$ and $M_0 = \langle C\rangle$. The boundary homomorphisms connect the chain groups into a sequence

$$0 \xrightarrow{D_3} M_2 \xrightarrow{D_2} M_1 \xrightarrow{D_1} M_0 \xrightarrow{D_0} 0.$$

We have

$$D_3 = \begin{bmatrix} 0 \end{bmatrix},\ D_2 = \begin{bmatrix} \langle \partial(ABC), BF\rangle \\ \langle \partial(ABC), CE\rangle \end{bmatrix},\ D_1 = \begin{bmatrix} \langle \partial(BF), C\rangle & \langle \partial(CE), C\rangle \end{bmatrix}, D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

To calculate $\langle \partial(ABC), BF\rangle$ we need to find paths from $\sigma \in \{AB, AC, BC\}$ to BF. For $\sigma = AB$ we have one path,

$$\alpha\colon AB,\ ABF,\ BF.$$

If we slide the edge AB along the path changing one vertex at the time while preserving the order, we get $AB \to FB$, so the final orientation does not match that of BF and the multiplicity of the path α is $m(\alpha) = -1$. For $\sigma = AC$ and $\sigma = BC$ there are no paths from σ to BF. So,

$$\langle \partial(ABC),\ BF\rangle = \langle \partial(ABC),\ AB\rangle \cdot m(\alpha) = \langle AB + BC - AC,\ AB\rangle \cdot (-1) = -1.$$

To calculate $\langle \partial(ABC), CE\rangle$ we need to find paths from $\sigma \in \{AB, AC, BC\}$ to CE. For $\sigma = AB$ and $\sigma = AC$, there are none. For $\sigma = BC$ we have one path,

$$\beta\colon BC,\ BCD,\ CD,\ CDE,\ CE.$$

Sliding BC along the path we get

$$BC \to DC \to EC,$$

so $m(\beta) = -1$ and

$$\langle \partial(ABC),\ CE\rangle = \langle \partial(ABC),\ BC\rangle \cdot m(\beta) = \langle AB + BC - AC,\ BC\rangle \cdot (-1) = -1.$$

To calculate $\langle \partial(BF), C\rangle$, we need to find paths from $\sigma\{B, F\}$ to C. For $\sigma = B$ we have just one path

$$\gamma\colon B,\ BD,\ D,\ DF,\ F,\ AF,\ A,\ AC,\ C.$$

The multiplicity of this path is $m(\gamma) = 1$. For $\sigma = F$ we also have a unique path

$$\delta\colon F,\ AF,\ A,\ AC,\ C,$$

again with multiplicity $m(\delta) = 1$. From here we can compute

$$\begin{aligned} \langle \partial(BF),\ C\rangle &= \langle \partial(BF),\ B\rangle m(\gamma) + \langle \partial(BF),\ F\rangle m(\delta) = \\ &= \langle F - B,\ B\rangle + \langle F - B,\ F\rangle = -1 + 1 = 0. \end{aligned}$$

To calculate $\langle \partial(CE),\ C\rangle$, we need to find paths from $\sigma\{C,\ E\}$ to C. For $\sigma = C$ we have just one (constant) path ϵ with $m(\epsilon) = 1$. For $\sigma = E$ we also have a unique path,

$$\theta\colon E,\ ED,\ D,\ DE,\ F,\ AF,\ A,\ AC,\ C,$$

with $m(\theta) = 1$. From here we can compute

$$\begin{aligned} \langle \partial(CE),\ C\rangle &= \langle \partial(CE),\ C\rangle m(\epsilon) + \langle \partial(CE),\ E\rangle m(\theta) = \\ &= \langle E - C,\ C\rangle + \langle E - C,\ E\rangle = -1 + 1 = 0. \end{aligned}$$

We have obtained

$$D_3 = \begin{bmatrix} 0 \end{bmatrix},\ D_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

We can now compute

$$\begin{aligned} H_2(M;\mathbb{Z}) &= \frac{\ker D_2}{\mathrm{im} D_3} = \frac{\langle 0\rangle}{\langle 0\rangle} = 0, \\ H_1(M;\mathbb{Z}) &= \frac{\ker D_1}{\mathrm{im} D_2} = \frac{\langle BF, CE\rangle}{\langle -BF - CE\rangle} = \langle BF\rangle = \mathbb{Z}, \\ H_0(M;\mathbb{Z}) &= \frac{\ker D_0}{\mathrm{im} D_1} = \frac{\langle C\rangle}{\langle 0\rangle} = \mathbb{Z}. \end{aligned}$$

Solution 60.

The given discrete vector field has one critical simplex in each of the dimensions 0, 1 and 2. So, $M_2 = \langle BCF \rangle$, $M_1 = \langle BE \rangle$ and $M_0 = \langle E \rangle$. The boundary homomorphisms connect the chain groups into a sequence

$$0 \xrightarrow{D_3} M_2 \xrightarrow{D_2} M_1 \xrightarrow{D_1} M_0 \xrightarrow{D_0} 0.$$

We have

$$D_3 = \begin{bmatrix} 0 \end{bmatrix},\ D_2 = \begin{bmatrix} \langle \partial(BCF), BE \rangle \end{bmatrix},\ D_1 = \begin{bmatrix} \langle \partial(BE), E \rangle \end{bmatrix}, D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

To compute $\langle \partial(BCF), BE \rangle$ we need to find all 2-paths that start at a simplex σ in the boundary of BCF and end at BE. There are three 1-dimensional simplices that are faces of BCE, namely BC, BF and CF, so $\sigma \in \{BC, BF, CF\}$. For $\sigma = CF$ we have a path

$$\alpha\colon CF,\ CDF,\ CD,\ ACD,\ AC,\ ACE,\ AE,\ AEF,\ AF,\ ABF,\ AB,\ ABD,\ BD,\ BDE,\ BE.$$

We need to determine the multiplicity of α, $m(\alpha)$. If we slide the edge CF along the path α changing one endpoint at the time keeping track of the orientation, we get

$$CF \to CD \to CA \to EA \to FA \to BA \to BD \to BE.$$

The final orientation matches, so $m(\alpha) = 1$. We get

$$\langle \partial(BCF),\ \sigma \rangle m(\alpha) = \langle BC + CF - BF,\ CF \rangle m(\alpha) = 1 \cdot 1 = 1.$$

For $\sigma = BC$ we find a path

$$\beta\colon BC,\ BCE,\ BE.$$

From $BC \to BE$ we conclude that $m(\beta) = 1$ and

$$\langle \partial(BCF),\ \sigma \rangle m(\beta) = \langle \partial(BCF),\ BC \rangle m(\beta) = 1 \cdot 1 = 1.$$

There are no 2-paths from $\sigma = BF$. We add up the results over all paths to get

$$\langle \partial(BCF),\ BE \rangle = 1 + 1 = 2.$$

To compute $\langle \partial(BE),\ E \rangle$ we need to find all 1-paths to E that start at the boundary of BE, so $\sigma \in \{B,\ E\}$. For $\sigma = B$ we have a path

$$\gamma\colon B,\ BF,\ F,\ DF,\ D,\ DE,\ E.$$

For 1-paths the multiplicity is always 1, so $m(\gamma) = 1$. We can now compute

$$\langle \partial(BE),\ \sigma \rangle m(\gamma) = \langle E - B,\ B \rangle m(\gamma) = -1 \cdot 1 = -1.$$

For $\sigma = E$ the only path δ from E to E is the constant path (it has to be, since the discrete Morse vector field is not supposed to contain loops). The multiplicity of the constant path is 1, so $m(\delta) = 1$. We get

$$\langle \partial(BE),\ \sigma \rangle m(\delta) = \langle E - B,\ E \rangle m(\delta) = 1 \cdot 1 = 1.$$

We add up the results over all possible paths to get

$$\langle \partial(BE), E \rangle = -1 + 1 = 0.$$

So,

$$D_3 = \begin{bmatrix} 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 2 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 \end{bmatrix},\ \text{and } D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

We can now compute

$$\begin{aligned} H_2(\mathbb{R}P^2;\mathbb{Z}) &= \frac{\ker D_2}{\mathrm{im} D_3} = \frac{\langle 0\rangle}{\langle 0\rangle} = 0,\\ H_1(\mathbb{R}P^2;\mathbb{Z}) &= \frac{\ker D_1}{\mathrm{im} D_2} = \frac{\langle BE\rangle}{\langle 2BE\rangle} = \mathbb{Z}_2,\\ H_0(\mathbb{R}P^2;\mathbb{Z}) &= \frac{\ker D_0}{\mathrm{im} D_1} = \frac{\langle E\rangle}{\langle 0\rangle} = \mathbb{Z}. \end{aligned}$$

If we used $\mathbb{Z}_2$ coefficients instead, we would get

$$D_3 = \begin{bmatrix} 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 \end{bmatrix},\ \text{and } D_0 = \begin{bmatrix} 0 \end{bmatrix},$$

so

$$\begin{aligned} H_2(\mathbb{R}P^2;\mathbb{Z}_2) &= \frac{\ker D_2}{\mathrm{im} D_3} = \frac{\langle BCF\rangle}{\langle 0\rangle} = \mathbb{Z}_2,\\ H_1(\mathbb{R}P^2;\mathbb{Z}_2) &= \frac{\ker D_1}{\mathrm{im} D_2} = \frac{\langle BE\rangle}{\langle 0\rangle} = \mathbb{Z}_2,\\ H_0(\mathbb{R}P^2;\mathbb{Z}_2) &= \frac{\ker D_0}{\mathrm{im} D_1} = \frac{\langle E\rangle}{\langle 0\rangle} = \mathbb{Z}_2. \end{aligned}$$

Solution 61.

The given discrete vector field has $c_2 = 1$, $c_1 = 2$ and $c_0 = 1$. The Morse complex is $M_2 = \langle CDF\rangle$, $M_1 = \langle BE,\, DH\rangle$ and $M_0 = \langle H\rangle$. The boundary homomorphisms connect the chain groups into a sequence

$$0 \xrightarrow{D_3} M_2 \xrightarrow{D_2} M_1 \xrightarrow{D_1} M_0 \xrightarrow{D_0} 0.$$

We have

$$D_3 = \begin{bmatrix} 0 \end{bmatrix},\ D_2 = \begin{bmatrix} \langle\partial(CDF), BE\rangle \\ \langle\partial(CDF), DH\rangle \end{bmatrix},\ D_1 = \begin{bmatrix} \langle\partial(BE),\, H\rangle \\ \langle\partial(DH),\, H\rangle \end{bmatrix}, D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

To compute $\langle\partial(CDF), BE\rangle$ we need to find all 2-paths that start at a simplex σ in the boundary of CDF and end at BE. There are three 1-dimensional simplices that are faces of CDF, namely CD, DF and CF, so $\sigma \in \{CD, DF, CF\}$. For $\sigma = CD$ we have a path

$$\alpha\colon CD,\, ACD,\, AC,\, ACI,\, AI,\, AGI,\, AG,\, ABG,\, AB,\, ABE,\, BE.$$

We need to determine the multiplicity of α, $m(\alpha)$. If we slide the edge CD along the path α changing one endpoint at the time keeping track of the orientation, we get

$$CD \to CA \to IA \to GA \to BA \to BE.$$

The final orientation matches, so $m(\alpha) = 1$. For $\sigma = DF$ there are no paths from DF to BE. For $\sigma = CF$ we have a path

$$\beta\colon CF,\, BCF,\, BF,\, BEF,\, BE.$$

The orientation changes are $CF \to BF \to BE$, so $m(\beta) = 1$. We have

$$\begin{aligned} \langle\partial(CDF), BE\rangle &= \langle CD + DF - CF,\, CD\rangle m(\alpha) + \langle CD + DF - CF,\, CF\rangle m(\beta) =\\ &= 1\cdot 1 + (-1)\cdot 1 = 1 - 1 = 0. \end{aligned}$$

To compute $\langle\partial(CDF), DH\rangle$ we need to find all 2-paths that start at $\sigma \in \{CD, DF, CF\}$ and end at DH. For $\sigma = CD$ we have a path

$$\gamma\colon CD,\, ACD,\, AC,\, ACI,\, AI,\, AGI,\, AG,\, ABG,\, AB,\, ABE,\, AE,\, ADE,\, DE,\, DEH,\, DH.$$

If we slide the edge CD along the path γ changing one endpoint at the time keeping track of the orientation, we get

$$CD \to CA \to IA \to GA \to BA \to EA \to ED \to HD.$$

The final orientation is reversed, so $m(\alpha) = -1$. We have another path for the same $\sigma = CD$, namely

$$\delta\colon CD,\ ACD,\ AC,\ ACI,\ AI,\ AGI,\ AG,\ ABG,\ BG,\ BGH,\ GH,\ DGH,\ DH.$$

The orientations are

$$CD \to CA \to IA \to GA \to GB \to GH \to DH.$$

The final orientation matches, so $m(\delta) = 1$. There are no paths from $\sigma = DF$ or $\sigma = CF$ to DH. We get

$$\begin{array}{rcl} \langle \partial(CDF), DH \rangle & = & \langle CD + DF - CF,\, CD \rangle (m(\gamma) + m(\delta)) \ = \\ & = & 1 \cdot (-1+1) \ = \ 0. \end{array}$$

Next, we need to count the number of 1-paths from $\partial(BE) = E - B$ to H. We have one path from E to H and one path from B to H, each with multiplicity 1, of course. So,

$$\langle \partial(BE),\, H \rangle = \langle E - B,\, E \rangle \cdot 1 + \langle E - B,\, B \rangle \cdot 1 = 1 - 1 = 0.$$

Similarly, for $\partial(DH) = H - D$ we have one 1-path from H to H (the constant path) and one 1-path from D to H. So,

$$\langle \partial(DH),\, H \rangle = \langle H - D,\, H \rangle \cdot 1 + \langle H - D,\, D \rangle \cdot 1 = 1 - 1 = 0.$$

This means that

$$D_3 = \begin{bmatrix} 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, D_0 = \begin{bmatrix} 0 \end{bmatrix}.$$

We have

$$\begin{array}{rcccccl} H_2(T;\mathbb{Z}) & = & \dfrac{\ker D_2}{\mathrm{im} D_3} & = & \dfrac{\langle CDF \rangle}{\langle 0 \rangle} & = & \mathbb{Z}, \\ H_1(T;\mathbb{Z}) & = & \dfrac{\ker D_1}{\mathrm{im} D_2} & = & \dfrac{\langle BE,\, DH \rangle}{\langle 0 \rangle} & = & \mathbb{Z} \oplus \mathbb{Z}, \\ H_0(T;\mathbb{Z}) & = & \dfrac{\ker D_0}{\mathrm{im} D_1} & = & \dfrac{\langle H \rangle}{\langle 0 \rangle} & = & \mathbb{Z}. \end{array}$$

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