Mathematical Modelling Exam

29. 6. 2020

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 105 minutes to solve the problems.

1. **[10 points]** Let A, B be $m \times n$ matrices, $m, n \in \mathbb{N}$, such that $A^T B = 0$ and $BA^T = 0$. Verify the following statements:

   Note: If you are not able to prove (1b), you can assume it is true in proving (1d), and also you can assume both of them are true in proving (1e).

   (a) **[1]** Every column of A is perpendicular to every column of B.

   Hint: What is the meaning of the entry in the i-th row and j-th column of $A^T B$?

   Solution.

$$\begin{aligned} 0 = (A^T B)_{i,j} &= \text{dot product of the } i\text{-th row of } A^T \text{ and } j\text{-th column of } B \\ &= \text{dot product of the } i\text{-th column of } A \text{ and } j\text{-th column of } B. \end{aligned}$$

   (b) **[2]** $A^+ B = B^+ A = 0$.

   Hint: Remember the geometric meaning of $A^+ b$ (resp. $B^+ a$), where b (resp. a) is a column in $\mathbb{R}^m$, and use this for every column of the matrix B (resp. A).

   Solution. $A^+ b$ is the vector with the smallest norm among all vectors from the set

$$\mathcal{S}(A, b) := \left\{ x \in \mathbb{R}^n \colon \|b - Ax\| = \min_{x' \in \mathbb{R}^n} \|b - Ax'\| \right\}.$$

   Since every column b_j of B is perpendicular to the span of the columns of A,

$$\min_{x' \in \mathbb{R}^n} \|b_j - Ax'\| = \|b_j\|$$

   and hence $0 \in \mathcal{S}(A, b_j)$. Thus $0 = A^+ b_j$.

   (c) **[1]** Every column of A^T is perpendicular to every column of B^T.

   Hint: What is the meaning of the entry in the i-th row and j-th column of $(B^T)^T A^T = BA^T$?

   Solution.

$$\begin{aligned} 0 = (BA^T)_{i,j} &= \text{dot product of the } i\text{-th row of } B \text{ and } j\text{-th column of } A^T \\ &= \text{dot product of the } i\text{-th column of } B^T \text{ and } j\text{-th column of } A^T. \end{aligned}$$

(d) **[2]** $BA^+ = AB^+ = 0$.

Hint: Assuming (1b) is true, this statement can be proved by plugging A^T and B^T into the appropriate variables in (1b).

Solution. Plugging A^T, B^T into A, B of (1b) we obtain

$$\begin{aligned} 0 &= (A^T)^+B^T = (A^+)^TB^T = (BA^+)^T \quad &\Rightarrow \quad 0 = BA^+, \\ 0 &= (B^T)^+A^T = (B^+)^TA^T = (AB^+)^T \quad &\Rightarrow \quad 0 = AB^+. \end{aligned}$$

(e) **[4]** $(A + B)^+ = A^+ + B^+$.

Hint: Use (1b), (1d) in the verification of this part.

Solution.

$$\begin{aligned} (A+B)(A^+ + B^+)(A+B) &= (AA^+ + AB^+ + BA^+ + BB^+)(A+B) \\ &= (AA^+ + BB^+)(A+B) \\ &= AA^+A + AA^+B + BB^+A + BB^+B \\ &= A^+ + 0 + 0 + B^+ \\ &= A^+ + B^+, \\ (A^+ + B^+)(A+B)(A^+ + B^+) &= (A^+A + A^+B + B^+A + B^+B)(A+B) \\ &= (A^+A + B^+B)(A^+ + B^+) \\ &= A^+AA^+ + A^+AB^+ + B^+BA^+ + B^+BB^+ \\ &= A + 0 + 0 + B \\ &= A + B, \\ ((A+B)(A^+ + B^+))^T &= (A^+ + B^+)^T(A+B)^T \\ &= (A^+)^TA^T + (A^+)^TB^T + (B^+)^TA^T + (B^+)^TB^T \\ &= (AA^+)^T + 0 + 0 + (BB^+)^T \\ &= AA^+ + BB^+, \\ ((A^+ + B^+)(A+B))^T &= (A+B)^T(A^+ + B^+)^T \\ &= A^T(A^+)^T + A^T(B^+)^T + B^T(A^+)^T + B^T(B^+)^T \\ &= (A^+A)^T + 0 + 0 + (B^+B)^T \\ &= A^+A + B^+B. \end{aligned}$$

2. **[10 points]** For the parametric curve

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2t - t^2 \\ 3t - t^3 \end{bmatrix}, \quad \text{where} \quad t \in \mathbb{R},$$

solve the following:

(a) **[1]** Find intersections with coordinate axes.

Solution.

$$\begin{aligned} x(t) = 0 \quad &\Leftrightarrow \quad 2t - t^2 = 0 \quad \Leftrightarrow \quad t(t-2) = 0 \quad \Leftrightarrow \quad t \in \{0, 2\}, \\ y(t) = 0 \quad &\Leftrightarrow \quad 3t - t^3 = 0 \quad \Leftrightarrow \quad t(3 - t^2) = 0 \quad \Leftrightarrow \quad t \in \{-\sqrt{3}, 0, \sqrt{3}\}. \end{aligned}$$

Intersections with y-axis: $(0,0)$, $(0,-2)$.
Intersections with x-axis: $(-2\sqrt{3}-3,0)$, $(0,0)$, $(2\sqrt{3}-3,0)$.

(b) **[1]** Find points at which the tangent is horizontal or vertical.

Solution.

$$\begin{aligned} \dot{x}(t)=0 \quad &\Leftrightarrow \quad 2-2t=0 \quad \Leftrightarrow \quad t=1, \\ \dot{y}(t)=0 \quad &\Leftrightarrow \quad 3-3t^2=0 \quad \Leftrightarrow \quad 1-t^2=0 \quad \Leftrightarrow \quad t\in\{-1,1\}. \end{aligned}$$

Horizontal tangent: $(-3,-2)$, vertical tangents: none.

(c) **[1]** Find points where $x'(t)=y'(t)=0$.

From the part above $(1,2)$.

(d) **[1]** Determine the asymptotic behaviour (limits as $t\to\pm\infty$).

Solution. $\lim\limits_{t\to-\infty} f(t)=\begin{bmatrix} -\infty \\ \infty \end{bmatrix}$, $\lim\limits_{t\to\infty} f(t)=\begin{bmatrix} -\infty \\ -\infty \end{bmatrix}$.

(e) **[2]** Show that there are no self-intersections.

Hint: To notice that the curve does not have any self-intersections verify that $1-x(t)=1-x(s)$ implies $s=2-t$ and plug this into the equation $y(t)=y(s)$.

Solution. Assume that $t\neq s$.

$$\begin{aligned} 1-x(t)=1-x(s) \quad &\Leftrightarrow \quad 1-2t+t^2=1-2s+s^2 \\ &\Leftrightarrow \quad (1-t)^2=(1-s)^2 \\ &\Leftrightarrow \quad 1-t\in\{1-s,s-1\}. \end{aligned}$$

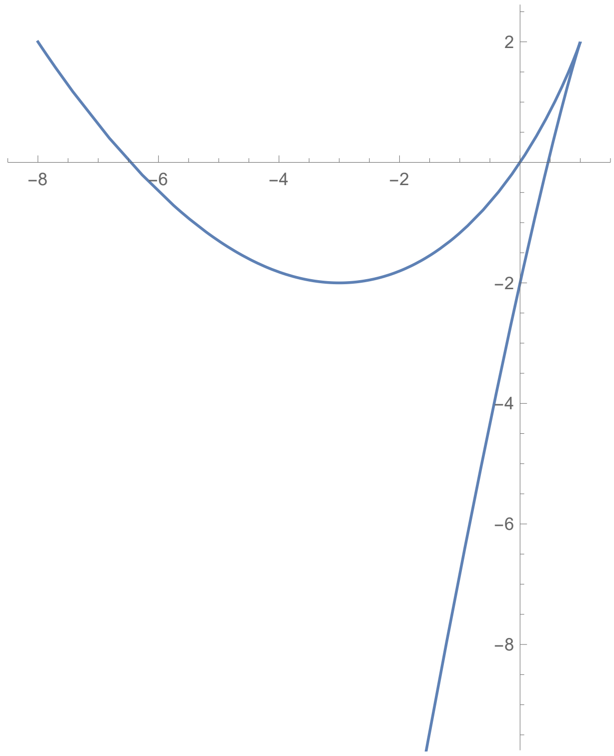
Since $t\neq s$, $1-t=s-1$ and hence $s=2-t$. Thus

$$\begin{aligned} y(t)=y(2-t) \quad &\Leftrightarrow \quad 3t-t^3=3(2-t)+(2-t)^3 \\ &\Leftrightarrow \quad t^3-3t^2+3t-1=0 \\ &\Leftrightarrow \quad (t-1)^3=0 \Leftrightarrow t=1. \end{aligned}$$

But then $s=2-1=1$ and $s=t$.

(f) **[4]** Plot the curve.

Solution. Using the information above, the sketch of the curve is the following:



3. **[10 points]** Let

$$F(x,y) := \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2+y^2-10x+y \\ x^2-y^2-x+10y \end{bmatrix}$$

be a vector function and $a=(2,4)\in\mathbb{R}^2$ a point.

(a) **[2]** Calculate the Jacobian matrix of the function F in the point a.

Solution.

$$JF(a) = \begin{bmatrix} 2x-10 & 2y+1 \\ 2x-1 & -2y+10 \end{bmatrix}(a) = \begin{bmatrix} -6 & 9 \\ 3 & 2 \end{bmatrix}.$$

(b) **[3]** Calculate the linear approximation of F in the point a.

Solution.

$$\begin{aligned} L_{F,a}(x,y) &= F(a) + JF(a)\begin{bmatrix} x-2 \\ y-4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 26 \end{bmatrix} + \begin{bmatrix} -6 & 9 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x-2 \\ y-4 \end{bmatrix} \\ &= \begin{bmatrix} -6x+9y-20 \\ 3x+2y+12 \end{bmatrix}. \end{aligned}$$

(c) **[5]** Perform one step of Newton's method to find the approximate solution of the system

$$F(x,y) = \begin{bmatrix} 1 \\ 25 \end{bmatrix}$$

with the initial approximation a.

Solution. We are searcing for zeroes of the vector function

$$G(x,y) = F(x,y) - \begin{bmatrix} 1 \\ 25 \end{bmatrix}$$

with the initial approximation $(x_0, y_0) = a$. One step of Newton's method:

$$\begin{aligned}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - JG(a)^{-1} G(x_0, y_0) \\
&= \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -6 & 9 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \frac{1}{39} \begin{bmatrix} 2 & -9 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
&= \frac{1}{13} \begin{bmatrix} 25 \\ 47 \end{bmatrix}.
\end{aligned}$$

4. **[10 points]** Find the solution $[x(t), y(t)]$ of the nonautonomous system of first order linear equations

$$\begin{aligned}
\dot{x} &= 2x - y, \\
\dot{y} &= -2x + y + 18t,
\end{aligned}$$

which satisfies $x(0) = 1$, $y(0) = 0$.

Hint: One of the particular solutions of the system above is of the form $x(t) = At^2 + Bt + C$, $y(t) = Dt^2 + Et + F$, where A, B, C, D, E, F are constants.

Solution. The system in the matricial form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 18t \end{bmatrix} =: \mathcal{A} \begin{bmatrix} x \\ y \end{bmatrix} + f(t).$$

Solution of the homogeneous part $(x_h(t), y_h(t))$:

$$\det(\mathcal{A} - \lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ -2 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - 2 = \lambda(\lambda - 3).$$

Hence $\det(\mathcal{A} - \lambda I) = 0$ for $\lambda_1 = 0$ and $\lambda_2 = 3$. Further on,

$$\begin{aligned}
\ker \mathcal{A} &= \ker \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \\
\ker(\mathcal{A} - 3I) &= \ker \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}.
\end{aligned}$$

Thus:

$$\begin{bmatrix} x_h(t) \\ y_h(t)) \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Particular solution $(x_p(t), y_p(t))$ using the hint:

$$\begin{aligned}
2At + B &= 2(At^2 + Bt + C) - (Dt^2 + Et + F) \\
&= (2A - D)t^2 + (2B - E)t + (2C - F), \\
2Dt + E &= -2(At^2 + Bt + C) + (Dt^2 + Et + F) + 18t \\
&= (-2A + D)t^2 + (-2B + E + 18)t + (-2C + F).
\end{aligned}$$

By comparing the coefficients at $1, t, t^2$ we get the system:

$2A - D = 0, \quad 2A = 2B - E, \quad B = 2C - F, \quad 2D = -2B + E + 18, \quad E = 2C - F,$

with a one parametric solution

$$(A, B, C, D, E, F) = (3, 2, C, 6, -2, -2 + 2C).$$

Choosing $C = 0$ we get

$$(x_p(t), y_p(t)) = (3t^2 + 2t, 6t^2 - 2t - 2).$$

Finally, a general solution is

$$\begin{bmatrix} x(t) \\ y(t)) \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3t^2 + 2t \\ 6t^2 - 2t - 2 \end{bmatrix}.$$

The one satisfying $x(0) = 1$, $y(0) = 0$, is $\alpha = 1$, $\beta = 0$.