Mathematical Modelling Exam

02. 06. 2021

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 105 minutes to solve the problems.

1. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{in} \quad b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

   (a) Find the Moore–Penrose inverse $A^\dagger$ of the matrix A.

   (b) Does the system $Ax = b$ have a solution? Why?

   (c) If the system is solvable, find the solution closest to the origin, if not, find the vector x^+ such that the error $\|Ax^+ - b\|$ is the smallest possible.

   (d) Find the Moore–Penrose inverse $(A^\dagger)^+$ of $A^\dagger$.

   Solution.

   (a) Since $\operatorname{rank} A = 2$, the matrix $A^T A \in \mathbb{R}^{2\times s}$ is invertible and hence $A^\dagger = \left(A^T A\right)^{-1} A^T$. So,

$$A^\dagger = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} \cdot A^T = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix}^{-1} \cdot A^T = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

   (b) If the systema $Ax = b$ has a solution, then one of the solutions is $x^+ = A^\dagger b$. So, we have to compute Ax^+ and see, if the result is b.

$$x^+ = A^\dagger b = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \Rightarrow \qquad Ax^+ = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \neq b.$$

   So the system $Ax = b$ does not have a solution.

   (c) The vector x^+ such that the error $\|Ax^+ - b\|_2$ is the smallest possible, is

$$A^\dagger b = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

   (d) The Moore–Penrose inverse of $A^\dagger$ is always A by the symmetry in A and $A^\dagger$ in the conditions the MP inverse must satisfy.

2. Given the parametric curve $\vec{r}(t) = (x(t), y(t)) = (t^3 - 4t, t^2 - 4)$.

   (a) find all points where it intersects the coordinate axes,

   (b) find the tangent to the curve at $t = 1$,

   (c) find the points on the curve where the tangent is horizontal or vertical,

(d) if there is a selfintersection, find it and compute the area inside the loop formed by the curve,

(e) sketch the curve.

Solution.

(a) The intersections with the x-axis correspond to $y(t) = 0$:

$$y(t) = 0 \quad \Leftrightarrow \quad t^2 - 4 = 0 \quad \Leftrightarrow \quad t = t_1 = -2,\ t = t_2 = 2.$$

Hence, there are two intersections with the x-axis:

$$P = \vec{r}(-2) = (0, 0), \qquad \vec{r}(2) = (0, 0).$$

The intersections with the y-axis correspond to $x(t) = 0$:

$$x(t) = 0 \quad \Leftrightarrow \quad t^3 - 4t = 0 \quad \Leftrightarrow \quad t(t^2 - 4) = 0 \quad \Leftrightarrow \quad t = t_1,\ t = t_2,\ t = t_3 = 0.$$

Hence, there are three intersections with the y-axis:

$$\vec{r}(-2) = (0, 0), \qquad \vec{r}(2) = (0, 0), \qquad Q = \vec{r}(0) = (0, -4).$$

(b) The tangent to the curve at the point $t = t_0$ is

$$\vec{s}(\lambda) = \vec{r}(t_0) + \lambda\, (\vec{r})'\, (t_0).$$

Hence,

$$\vec{s}(\lambda) = (-3, -3) + \lambda(3t^2 - 4, 2t)(1) = (-3, -3) + \lambda(-1, 2) = (-3 - \lambda, -3 + 2\lambda).$$

(c) The tangent to the curve is horizontal at the points, where $y'(t) = 0$. Hence,

$$y'(t) = 0 \quad \Leftrightarrow \quad 2t = 0 \quad \Leftrightarrow \quad t = 0,$$

and the point is $Q = (0, -4)$.

The tangent to the curve is vertical at the points, where $x'(t) = 0$. Hence,

$$x'(t) = 0 \quad \Leftrightarrow \quad 3t^2 - 4 = 0 \quad \Leftrightarrow \quad t_4 = \frac{2}{\sqrt{3}},\ t_5 = -\frac{2}{\sqrt{3}},$$

and the points are

$$R = \left(-\frac{16}{3\sqrt{3}}, -\frac{8}{3}\right) \approx (3.079, -2.67), \qquad S = \left(\frac{16}{3\sqrt{3}}, -\frac{8}{3}\right) \approx (-3.079, -2.67).$$

(d) Self-intersections correspond to different values of t_1, t_2, where $\vec{r}(t_1) = \vec{r}(t_2)$:

$$\begin{aligned}
\vec{r}(t_1) = \vec{r}(t_2) \quad &\Leftrightarrow \quad t_1^3 - 4t_1 = t_2^3 - 4t_2 \quad \text{and} \quad t_1^2 - 4 = t_2^2 - 4 \\
&\Leftrightarrow \quad t_1^3 - 4t_1 = t_2^3 - 4t_2 \quad \text{and} \quad t_1^2 = t_2^2 \\
&\Leftrightarrow \quad t_1^3 - 4t_1 = (-t_1)^3 + 4t_1 \quad \text{and} \quad t_2 = -t_1 \\
&\Leftrightarrow \quad 2t_1(t_1^2 - 4) = 0 \quad \text{and} \quad t_2 = -t_1 \\
&\Leftrightarrow \quad t_1 = 2, t_2 = -2 \quad \text{or} \quad t_1 = -2, t_2 = 2.
\end{aligned}$$

Hence, the only self-intersection is the point P.

The area A inside the loop formed by the curve between $t_1 = -2$ and $t_2 = 2$ is

$$\begin{aligned} A &= \frac{1}{2}\int_{-2}^{2} \left|(t^3-4t)2t-(t^2-4)(3t^2-4)\right|\ dt \\ &= \frac{1}{2}\int_{-2}^{2} \left|-t^4+8t^2-16\right|\ dt. \end{aligned}$$

Since

$$t^4-8t^2+16=(t^2-4)^2=(t-2)^2(t+2)^2,$$

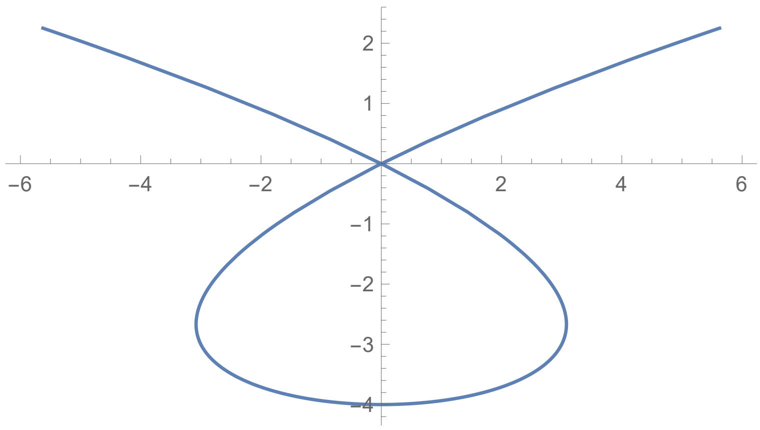
we have that

$$\left|-t^4+8t^2-16\right| = t^4-8t^2+16$$

for every $t \in \mathbb{R}$. Hence,

$$A = \frac{1}{2}\int_{-2}^{2}\left(t^4-8t^2+16\right)\ dt = \frac{1}{2}\left[\frac{t^5}{5}-8\frac{t^3}{3}+16t\right]_{-2}^{2} = \frac{512}{30}.$$

(e) The sketch of the curve is the following:



3. Find the general solution to the differential equation

$$y' = 2x(1+y^2)$$

and the particular solution that satisfies $y(1) = 0$.

Solution. The DE can be solved by separation of variables:

$$\frac{dy}{1+y^2} = 2x\ dx \quad \Rightarrow \quad \arctan y = x^2 + C.$$

The particular solution, which goes through the point $(1,0)$, is

$$\arctan y(1) = \arctan(0) = 0 = 1^2 + C \quad \Rightarrow \quad C = -1 \quad \Rightarrow \quad \arctan y = x^2 - 1.$$

4. For the system of nonlinear differential equations

$$\dot{x} = xy + 1, \quad \dot{y} = x + xy$$

(a) find its stationary point,

(b) classify it as a saddle, source, sink or center,

(c) sketch the phase portrait of the system around the stationary point.

Solution.

(a) Stationary points satisfy $\dot{x} = \dot{y} = 0$. Hence,

$$\begin{aligned} \dot{x} = xy + 1 = 0 \quad \text{and} \quad \dot{y} = x + xy = 0 \quad &\Leftrightarrow \quad xy = -1 \quad \text{and} \quad x = 1 \\ &\Leftrightarrow \quad y = -1 \quad \text{and} \quad x = 1. \end{aligned}$$

(b) To classify the stationary point $(1, -1)$ we have to linearize the system. We denote the right side of the system by

$$f(x, y) = (xy + 1, x + xy).$$

Hence,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &\approx ((Jf)(1, -1)) \cdot \begin{bmatrix} x - 1 \\ y + 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} y & x \\ 1 + y & x \end{bmatrix} (1, -1) \right) \cdot \begin{bmatrix} x - 1 \\ y + 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y + 1 \end{bmatrix} \\ &= \begin{bmatrix} -x + y + 2 \\ y + 1 \end{bmatrix}. \end{aligned}$$

The behaviour of the system depends on the eigenvalues of $(Jf)(1, -1)$:

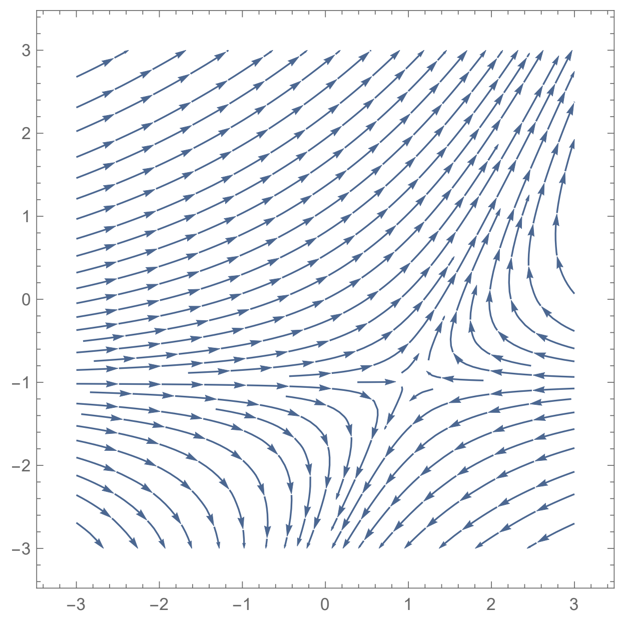
$$\det\left((Jf)(1, -1) - \lambda I_2\right) = \begin{bmatrix} -1 - \lambda & 1 \\ 0 & 1 - \lambda \end{bmatrix} = (-1 - \lambda)(1 - \lambda).$$

Hence, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. The solution of the system is of the form

$$C \cdot e^t \cdot v_1 + D \cdot e^{-t} \cdot v_2 + \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where v_1, v_2 are the eigenvectors of $(Jf)(1, -1)$ for the eigenvalues $1, -1$ and C, D are constants. The point $(1, -1)$ is a saddle.

(c) The sketch of the phase portrait is the following:



3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3