# Mathematical Modelling Exam

17. 6. 2020

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 105 minutes to solve the problems.

1. (a) Prove that if $AA^\mathsf{T}$ is an invertible matrix, then $A^\mathsf{T}(AA^\mathsf{T})^{-1}$ is the Moore–Penrose inverse $A^+$ of the matrix $A$ (i.e. check that it satifies all four requirements for $A^+$).

   (b) Find the point on the intersection of the planes $x+y+z=0$ and $x-y=1$ that is closest to the origin:

      i. write down the matrix of the system for the intersection and find its Moore-Penrose inverse,

      ii. among all solutions of the system find the one closest to the origin.

*Solution:*

(a) We denote $G := A^T(AA^T)^{-1}$. There are four requirements to check for $G$ to be equal to $A^\dagger$:

$$
\begin{aligned}
AGA &= A\left(A^T\left(AA^T\right)^{-1}\right)A = \left(AA^T\right)\left(AA^T\right)^{-1}A = IA = A,\\
GAG &= GA\left(A^T\left(AA^T\right)^{-1}\right) = G\left(A^TA\right)\left(A^TA\right)^{-1} = G,\\
(AG)^T &= \left(A\left(A^T\left(AA^T\right)^{-1}\right)\right)^T = \left(\left(AA^T\right)\left(AA^T\right)^{-1}\right)^T = I = AG,\\
(GA)^T &= \left(\left(A^T\left(AA^T\right)^{-1}\right)A\right)^T = \left(A^T\left(\left(AA^T\right)^{-1}\right)^T\left(A^T\right)^T\right) = GA,
\end{aligned}
$$

where we used that $\left(A^TA\right)^{-1}$ is symmetric in the last equality.

(b) i. The matricial form of the system is the following:

$$
\underbrace{\begin{bmatrix} 1 & 1 & 1\\ 1 & -1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0\\ 1 \end{bmatrix}}_{b}.
$$

The Moore-Penrose inverse of $A$ is the following:

$$
A^\dagger = A^T(AA^T)^{-1} = \begin{bmatrix} 1 & 1\\ 1 & -1\\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0\\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2}\\ \frac{1}{3} & -\frac{1}{2}\\ \frac{1}{3} & 0 \end{bmatrix}.
$$

ii. The solution of the system of the smallest norm is

$$
x^+ = A^\dagger b = \begin{bmatrix} \frac{1}{3} & \frac{1}{2}\\ \frac{1}{3} & -\frac{1}{2}\\ \frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\\ -\frac{1}{2}\\ 0 \end{bmatrix}.
$$

2. For the curves given in polar coordinates by $r = 2\sin\varphi$ and $r = 2\cos\varphi$:

   (a) prove that both curves are circles and provide a sketch (hint: try multiplying the equation by $r$ and expressing $r^2$, $r\sin\varphi$ and $r\cos\varphi$ by $x$ and $y$),

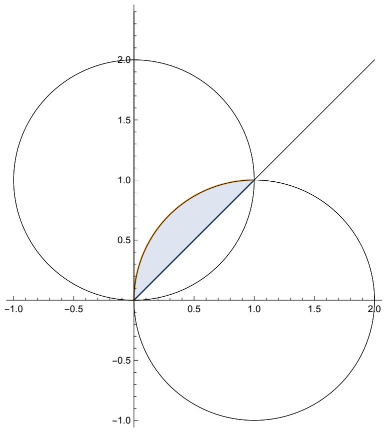
   (b) compute the area of the region that lies inside both circles.

*Solution:*

(a) We have

$$\begin{aligned} r_1^2 = r_1 \cdot 2\sin\varphi \quad &\Rightarrow \quad x^2 + y^2 = 2y \quad \Leftrightarrow \quad x^2 + (y-1)^2 = 1, \\ r_2^2 = r_2 \cdot 2\cos\varphi \quad &\Rightarrow \quad x^2 + y^2 = 2x \quad \Leftrightarrow \quad (x-1)^2 + y^2 = 1, \end{aligned}$$

where we used $r_i^2 = x^2 + y^2$, $r_i \cos\varphi = x$ and $r_i \sin\varphi = y$, $i = 1, 2$. The first equation $r_1(\varphi)$ is the equation of the circle with center at $(0, 1)$ and radius 1, and the second $r_2(\varphi)$ is the circle with center at $(1, 0)$ and radius 1. The sketch is the following




(b) The area can be computed using elementary geometry or using the formula for integrals of regions in polar coordinates. If $A$ is the shaded region in the plot, then

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\varphi)^2 \, d\varphi = \int_{\pi/4}^{\pi/2} (1 + \cos(2\varphi)) \, d\varphi \\ &= \left[\varphi + \frac{\sin(2\varphi)}{2}\right]_{\pi/4}^{\pi/2} = \frac{\pi}{4} - \frac{1}{2}, \end{aligned}$$

and the total area is $2A = \dfrac{\pi}{2} - 1$.

3. Given the differential equation $y' = 2xy^2$ with initial condition $y(0) = 1$

   (a) find its exact solution,

   (b) use Euler's method with step size 0.2 to estimate $y(0.4)$ and compare the result to the exact value $y(0.4)$.

*Solution:*

(a) The exact solution can be obtained using separation of variables:

$$\frac{dy}{dx} = 2xy^2 \quad \Rightarrow \quad \frac{dy}{y^2} = 2x\,dx \quad \Rightarrow \quad -\frac{1}{y} = x^2 + C \quad \Rightarrow \quad y = -\frac{1}{x^2+C}.$$

The solution passing through the point $(0,1)$ is obtained by

$$y(0) = 1 = -\frac{1}{C} \quad \Rightarrow \quad C = -1 \quad \Rightarrow \quad y = \frac{1}{1-x^2}.$$

(b) Using Euler's method to estimate $y(0.4)$ we get:

$$\begin{aligned} y_1 &= y(0.2) = 1 + 0.2 \cdot (0 \cdot 1^2) = 1, \\ y_2 &= y(0.4) = 1 + 0.2 \cdot (2 \cdot 0.2 \cdot 1^2) = 1.08. \end{aligned}$$

The exact solution is

$$y(0.4) = \frac{1}{1-0.16} \approx 1.2.$$

4. Find the general solution of the nonhomogeneous second order linear equation $\ddot{x} + \dot{x} - 2x = t^2$.

*Solution:* First we solve the homogenenous part

$$\ddot{x} + \dot{x} - 2x = 0.$$

The characterstic polynomial is $p(\lambda) := \lambda^2 + \lambda - 2$ and hence

$$p(\lambda) = 0 \quad \Leftrightarrow \quad (\lambda - 2)(\lambda + 1) = 0.$$

So, the solution of the homogeneous part is

$$x_h(t) = Ce^{2t} + De^{-t},$$

where $C, D \in \mathbb{R}$ are constants. To find one particular solution we use the form

$$x_p = At^2 + Bt + C.$$

Hence, $\dot{x}_p = 2At + B$ and $\ddot{x}_p = 2A$. Plugging this into the DE we obtain

$$2A + (2At + B) - 2\left(At^2 + Bt + C\right) = t^2. \tag{1}$$

Comparing the coefficients at $1, t, t^2$ on both sides of (1) we get the system

$$2A + B - 2C = 0, \qquad 2A - 2B = 0, \qquad -2A = 1,$$

with the solution

$$A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{3}{4}.$$

Hence, the general solution of the DE is

$$x(t) = Ce^{2t} + De^{-t} - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}.$$

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