Mathematical Modelling Exam

16. 08. 2021

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Perform one step of Gauss-Newton method to approximate the least squares solution of the system
$$f(x,y) = (2,3,1),$$
where
$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) = (x^2 + y^3 + 2, x + e^{y-1}, \sin x + 2y^2 - 3).$$
For the initial approximation take $(x_0, y_0) = (0,1)$.

2. Let S be a surface given by $z = g(x,y)$, where
$$g(x,y) = x^3 - x^2 y + y^2 - 2x + 3y - 2$$
is a differentiable function. Determine the tangent plane to S in the point $(-1,3)$ in the parametric and implicit form.

   Hint: Note that the parametric equation $f : \mathbb{R}^2 \to \mathbb{R}^3$ of the surface S is
$$f(x,y) = (x, y, g(x,y)).$$

3. Let
$$2xy - 9x^2 + (2y + x^2 + 1) \cdot \frac{dy}{dx} = 0 \tag{1}$$
be a differential equation.
   (a) Rewrite (1) in the form $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$ and prove that this DE is exact by checking the necessary and sufficient condition involving partial derivatives of M and N.
   (b) Solve the DE (1) with an initial condition $y(0) = -3$.

4. Convert the differential equation
$$y'' + 11y' + 24y = 0 \tag{2}$$
into the system of first order DEs, solve this system and recover the solution of the initial DE (2).