Mathematical Modelling Exam

16. 08. 2021

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Perform one step of Gauss-Newton method to approximate the least squares solution of the system
$$f(x,y) = (2,3,1),$$
where
$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) = (x^2 + y^3 + 2, x + e^{y-1}, \sin x + 2y^2 - 3).$$
For the initial approximation take $(x_0, y_0) = (0,1)$.

Solution. We define a vector function $F : \mathbb{R}^2 \to \mathbb{R}^3$ by the rule
$$F(x,y) := f(x,y) - (2,3,1) = (x^2 + y^3, x + e^{y-1} - 3, \sin x + 2y^2 - 4).$$
After one step of Gauss-Newton method the approximate of the least squares solution of the system $F(x,y) = (0,0,0)$ will be
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - ((JF)(x_0,y_0))^\dagger F(x_0,y_0).$$
We have
$$(JF)(x,y) = \begin{bmatrix} 2x & 3y^2 \\ 1 & e^{y-1} \\ \cos x & 4y \end{bmatrix} \qquad \Rightarrow \qquad (JF)(0,1) = \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}.$$
Since
$$((JF)(0,1))^T \cdot (JF)(0,1) = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 26 \end{bmatrix}$$
is invertible, it follows that
$$\begin{aligned} ((JF)(0,1))^\dagger &= \left(((JF)(0,1))^T \cdot (JF)(0,1) \right)^{-1} ((JF)(0,1))^T \\ &= \begin{bmatrix} 2 & 5 \\ 5 & 26 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \\ &= \frac{1}{27} \begin{bmatrix} 26 & -5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \\ &= \frac{1}{27} \begin{bmatrix} -15 & 21 & 6 \\ 6 & -3 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{9} & \frac{7}{9} & \frac{3}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \end{bmatrix} \end{aligned}$$
and
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{5}{9} & \frac{7}{9} & \frac{3}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{23}{9} \\ \frac{7}{9} \end{bmatrix}.$$

2. Let S be a surface given by $z = g(x, y)$, where

$$g(x, y) = x^3 - x^2 y + y^2 - 2x + 3y - 2$$

is a differentiable function. Determine the tangent plane to S in the point $(-1, 3)$ in the parametric and implicit form.

Hint: Note that the parametric equation $f : \mathbb{R}^2 \to \mathbb{R}^3$ of the surface S is

$$f(x, y) = (x, y, g(x, y)).$$

Solution. The parametric form $r : \mathbb{R}^2 \to \mathbb{R}^3$ of the tangent plane to S in the point $(x, y) = (-1, 3)$ is

$$\begin{aligned} r(u, v) &= f(-1, 3) + u f_x(-1, 3) + v f_y(-1, 3) \\ &= (-1, 3, g(-1, 3)) + u \cdot (1, 0, g_x(-1, 3)) + v \cdot (0, 1, g_y(-1, 3)). \end{aligned}$$

We have

$$\begin{aligned} g(-1, 3) &= (-1)^3 - (-1)^2 \cdot 3 + 3^2 - 2(-1) + 3 \cdot 3 - 2 = 14, \\ g_x(x, y) &= 3x^2 - 2xy - 2 \quad \Rightarrow \quad g_x(-1, 3) = 3 - 2(-1)3 - 2 = 7, \\ g_y(x, y) &= -x^2 + 2y + 3 \quad \Rightarrow \quad g_y(-1, 3) = -1 + 2 \cdot 3 + 3 = 8. \end{aligned}$$

Hence,

$$r(u, v) = (-1, 3, 14) + u(1, 0, 7) + v(0, 1, 8) = (-1 + u, 3 + v, 14 + 7u + 8v).$$

For the implicit form of the tangent plane we need its normal

$$\vec{n} := f_x(-1, 3) \times f_y(-1, 3) = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} -7 \\ -8 \\ 1 \end{bmatrix}.$$

Hence, the implicit form

$$(x, y, z) \cdot \vec{n} = f(-1, 3) \cdot \vec{n}$$

is

$$-7x - 8y + z = 7 - 24 + 14 = -3.$$

3. Let

$$2xy - 9x^2 + (2y + x^2 + 1) \cdot \frac{dy}{dx} = 0 \tag{1}$$

be a differential equation.

(a) Rewrite (1) in the form $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$ and prove that this DE is exact by checking the necessary and sufficient condition involving partial derivatives of M and N.

(b) Solve the DE (1) with an initial condition $y(0) = -3$.

Solution.

(a) An equivalent form of the DE (1) is

$$\underbrace{(2xy - 9x^2)}_{M(x,y)}\,\mathrm{d}x + \underbrace{(2y + x^2 + 1)}_{N(x,y)}\,\mathrm{d}y = 0.$$

Since $M(x, y)$ and $N(x, y)$ are differentiable for every $(x, y) \in \mathbb{R}^2$, the DE (1) is exact if and only if

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y).$$

We have

$$\frac{\partial M}{\partial y}(x, y) = 2x = \frac{\partial N}{\partial x}(x, y)$$

and hence the DE (1) is exact.

(b) Solutions of the exact DE are level curves $u(x, y) = C$, $C \in \mathbb{R}$, of the potential function $u(x, y)$, i.e., a function which satisfies

$$\frac{\partial u}{\partial x}(x, y) = M(x, y) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x, y) = N(x, y).$$

Hence,

$$u(x, y) = \int M(x, y)\mathrm{d}x = x^2y - 3x^3 + C(y),$$
$$u(x, y) = \int N(x, y)\mathrm{d}y = y^2 + x^2y + y + D(x),$$

where $C(y)$ and $D(x)$ are functions of y and x, respectively. So,

$$u(x, y) = x^2y - 3x^3 + y^2 + y + E,$$

where $E \in \mathbb{R}$ is a constant. The level curves of $u(x, y)$ are given by the equations

$$x^2y - 3x^3 + y^2 + y = C, \quad C \in \mathbb{R}. \tag{2}$$

The solution satisfying $y(0) = -3$ is the level curve (2)

$$0^2 \cdot (-3) - 3 \cdot 0^3 + (-3)^2 + (-3) = 6 = C$$

with $C = 6$.

4. Convert the differential equation

$$y'' + 11y' + 24y = 0 \tag{3}$$

into the system of first order DEs, solve this system and recover the solution of the initial DE (3).

Solution. We introduce functions $x_1(t) := y(t)$ and $x_2(t) := x_1'(t)$. The DE (3) converts into the system

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -24x_1(t) - 11x_2(t). \end{aligned} \tag{4}$$

The matricial form of (4) is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -24 & -11 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

The eigenvalues of the matrix A are the roots of the following determinant

$$\det\left(A - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = (-\lambda)(-11-\lambda) + 24 = \lambda^2 + 11\lambda + 24 = (\lambda+8)(\lambda+3).$$

So, $\lambda_1 = -8$ and $\lambda_2 = -3$.

The kernel of

$$A - \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -24 & -3 \end{bmatrix}$$

contains the vector

$$u_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}.$$

The kernel of

$$A - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -24 & -8 \end{bmatrix}$$

contains the vector

$$u_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

So, the general solution of the system (4) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cdot e^{-8t} \cdot \begin{bmatrix} 1 \\ -8 \end{bmatrix} + C_2 \cdot e^{-3t} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

where C_1 and C_2 are constants. Hence, the solution of the initial DE is

$$y(t) = Ce^{-8t} + De^{-3t},$$

where C, D are constants.