Mathematical Modelling Exam

June 5th, 2024

You have 90 minutes to solve the problems. The numbers in [$\cdot$] represent points.

1. Answer the following questions. In YES/NO questions **verify your reasoning**.

   (a) **[2]** Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ a vector. The orthogonal projection of b to the linear span of the columns of A is equal to AA^+b. YES/NO

   (b) **[2]** Assume that the Gauss-Newton method used to solve an overdetermined system $f(x) = 0$ converges to $\tilde{x} \in \mathbb{R}^n$ for some initial approximation $x^{(0)}$. Then $\tilde{x}$ is a least squares error solution to the system. YES/NO

   (c) **[2]** Let $\mathcal{C} = \{(x(t), y(t)) : t \in \mathbb{R}\}$ be some curve in the xy–plane. Let $\mathcal{S}$ be a surface obtained by revolving $\mathcal{C}$ around x–axis for 360 degrees. Write down the parametrization of $\mathcal{S}$.

   (d) **[2]** For every choice of the constants $c_i \in [0,1]$, $a_{ij} \in [0,\infty)$ and $b_i \in [0,1]$, the Runge–Kutta method with a Butcher tableau

$$\begin{array}{c|cccc} 0 & 0 & & & \\ c_2 & a_{21} & 0 & & \\ c_3 & a_{31} & a_{32} & 0 & \\ c_4 & a_{41} & a_{42} & a_{43} & 0 \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array}$$

   for solving the differential equation $y'(x) = f(x, y)$, $y(x_0) = y_0$ will be of order 4. YES/NO

   (e) **[2]** Let

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= (*), \\ \dot{x}_3 &= (*), \\ \dot{x}_4 &= -3x_1 + x_2 + 4x_3 + 5x_4, \end{aligned}$$

   by a system of differential equations, which comes in a standard way from some higher order differential equation with one dependent variable. What are $(*)$ and what was the original differential equation?

2. We are given points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$. We would like to determine α and β, such that the function

$$f(x) = \cos\alpha \cdot e^{\beta x}$$

   fits best to the points in the sense of least squares error.

   (a) **[3]** Write down explicitly the nonlinear system we have to solve. Identify the variables.

   (b) **[4]** Write one step of Gauss–Newton method for solving the problem. Determine the Jacobian matrix needed explicitly.

(c) **[3]** Compute $J^T J$ explicitly, where J is the Jacobian from the previous question and comment on the efficient way of computing J^+. You do not need to compute J^+ explicitly.

3. Let

$$x(t) = \sin^2 t, \quad y(t) = 2\cos t, \quad t \in \mathbb{R}$$

be a curve.

(a) **[7]** Sketch the curve. Determine also all local extrema in x and y direction and all intersections with the axes.

(b) **[8]** Compute the length of the trace of the curve.

Hint: Use $\int \sqrt{u^2+1}du = \frac{1}{2}\log(u+\sqrt{1+u^2}) + \frac{1}{2}u\sqrt{1+u^2} + C$, $C \in \mathbb{R}$.

4. Let $x^2 - y^2 = ay$, $a \in \mathbb{R}$, be a family of curves.

(a) **[5]** Plot a few members of the family.

(b) **[5]** Derive a differential equation determining orthogonal trajectories to the given family.

(c) **[5]** Solve the differential equation obtained.