Mathematical Modelling Exam

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. (a) Compute the truncated singular value decomposition of the matrix

$$A = \begin{pmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

   (b) Write down a matrix B with 6 rows and 3 columns, such that none of the columns consists of only 0-s, and such that B has the same left singular vectors as A and the same nonzero singular values as A. Justify your answer.

2. Let

$$\sin(\pi x)\cos(\pi y) = \cos(\pi x),$$
$$\sin(\pi x)\sin(\pi y) = \frac{1}{2}$$

   be a nonlinear system and $v^{(0)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \end{bmatrix}^T$ a vector. Compute the approximation $v^{(1)}$ of the solution to the system using one step of Newton's method.

3. Let

$$f(\phi, \psi) = \begin{pmatrix} \cos\phi \\ \sin\phi\cos\psi \\ \sin\phi\sin\psi \end{pmatrix}, \qquad \phi \in [0, \pi], \ \psi \in [0, 2\pi]$$

   be a parametrization of some surface S.

   (a) Which surface is S?
   (b) Compute the Jacobian matrix $J_f(\phi, \psi)$ of $f(\phi, \psi)$.
   (c) Compute $\det\left((J_f(\phi, \psi))^T J_f(\phi, \psi)\right)$.

   (d) The surface area of a slice of S for $\phi \in [0, \pi]$ and $\psi \in \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$ is equal to

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\psi \int_0^{\pi} \sqrt{\det\left((J_f(\phi, \psi))^T J_f(\phi, \psi)\right)}\, d\phi.$$

   Compute this surface area and denote on a sketch of S, what part of the surface this represents.

4. Solve the differential equation

$$y''(t) + 2y'(t) + y(t) = 2\sin(t) + t^2 + 9e^{2t}, \qquad y(0) = 15,\ y'(0) = 14.$$