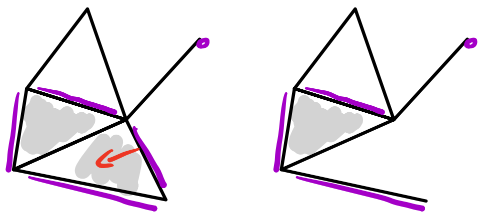
# Discrete Morse Theory

## ① Motivation (An easy simplification of a scx)

**Def:** • $K$ scx. $\sigma \in K$ is a free face if it is a face of exactly one sx $\tau \in K$.

$\tau$ is a max face, $\dim \tau = \dim \sigma + 1$

free faces



elementary collapse

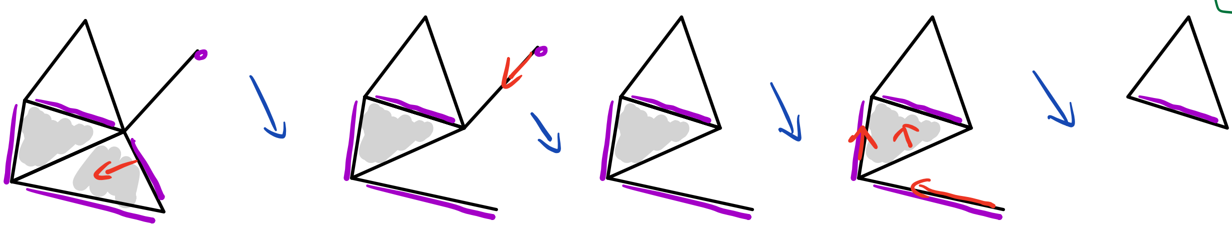
• An elementary collapse is a removal $K \rightsquigarrow K \setminus \{\sigma, \tau\}$. In this case $K \setminus \{\sigma,\tau\} \hookrightarrow K$ is a homotopy equivalence.

• Scx $K$ is collapsible $[K \searrow L]$ to a sub scx $L \subseteq K$ if there exists a collapse (i.e., a sequence of elementary collapses) reducing $K$ to $L$.

• $K$ is collapsible if $K \searrow \cdot$.

$\| \quad K \searrow L \overset{\not\Leftarrow}{\Rightarrow} K \simeq L$

Dunce hat
Bing's house



<u>IDEA:</u> let's try to formalize collapsing sequence.

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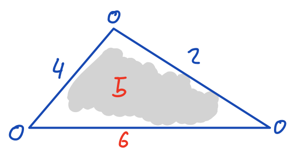
## ② Discrete Morse functions [DMF] and discrete vector fields [DVF]

**Def:** $K$ scx. A function $f: K \to \mathbb{R}$ is a DMF if $\forall \sigma^k \in K$:



ⓐ $e_1 = |\{\tau^{k-1} < \sigma \,;\, f(\tau) \geq f(\sigma)\}| \leq 1$, and

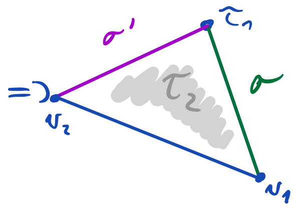
ⓑ $e_2 = |\{\tau^{k+1} > \sigma \,;\, f(\tau) \leq f(\sigma)\}| \leq 1$.

$\underbrace{\left.\begin{array}{l}\text{faces}\\ \text{cofaces}\end{array}\right\} \text{where } f \text{ does not abide by dim.}}_{\text{exceptional face or coface}}$

<u>Prop:</u> $e_1 \cdot e_2 = 0$.

<u>Proof:</u> Assume $\tau_1^{k-1} \in \sigma$ is exceptional for $\sigma$

$\tau_2^{k+2} \geq \sigma$ is exceptional for $\sigma$



$f(\tau_1) > f(\sigma) > f(\tau_2)$

$f(\tau_1) < f(\sigma') < f(\tau_2)$

$\rightarrow\leftarrow$

<u>Cor:</u> $\sigma$ is an except. face of $\tau$ iff $\tau$ is an except. coface of $\sigma$.

Pairs (sx, except. face) are disjoint.

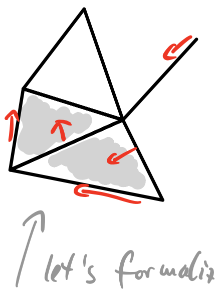
**Def:** Given a DMF $f$ on a sex $K$, a pair ($\text{sx } \sigma$, except. face of $\sigma$) is called a **regular pair**.

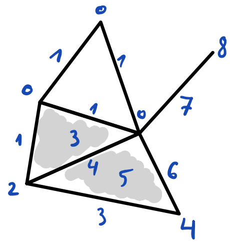
Sxes of $K$ are partitioned into: → regular pairs (indicated by arrows).

→ critical sxes (where $f$ completely respects dim.)

(fewer crit. sxes, better simplification)

**Example:** Collapse is induced by a DMF (not uniquely).




**Prop:** $n_i$ .... # of critical $n$-sxes

Then $\chi = n_0 - n_1 + n_2 - \dots$

**Proof:** removing a regular pair preserves $\chi$. ■

**Def:** $K$ sex. A **discrete vector field** [DVF] on $K$ is a disjoint collection of pairs $\underbrace{(\sigma_i, \tau_i)}$ of sxes of $K$, such that $\sigma_i$ is a face of $\tau_i$, $\forall i$. Critical sxes NOT INVOLVED.

↳ such pairs are called arrows.

A DVF is called a **discrete gradient vector field** [DGVF] if it is induced by a DMF (as a collection of regular pairs).

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## ③ DGVF's (recognizing DGVF's)

**Def:** $K$ sex, $p \in \mathbb{N}$. Given a DVF on $K$ consisting of pairs $\{(\sigma_j, \tau_j)\}_{j \in J}$, a **p-path** is a sequence

$$\underbrace{\sigma_{j_1}^{p-1} \to \tau_{j_1}^{p}}_{\uparrow} \geq \underbrace{\overset{\swarrow \text{ sup sx}}{\sigma_{j_2}^{p-1}} \to \tau_{j_2}^{p}}_{\nearrow} \geq \sigma_{j_3}^{p-1} \to \dots \to \tau_{j_K}^{p} \geq \sigma_{K+1}^{p-1}$$

arrows in a DVF

Such a path is a **cycle** if $\sigma_1 = \sigma_{K+1}$ and $K \geq 1$.

A DVF is **acyclic** if it admits no cycle.

**Observations:** ⓐ a crit. sx can only appear as the last sx in a p-path

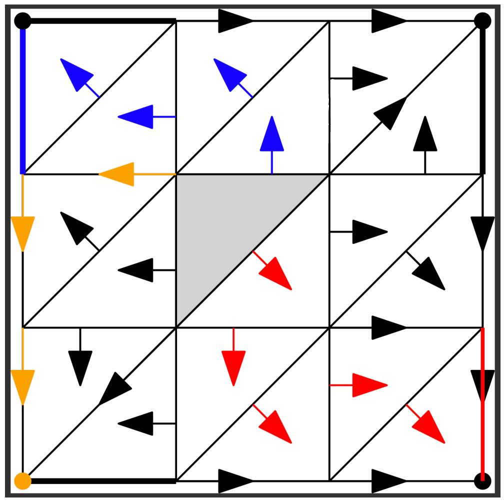
ⓑ Given a DMF $f$, function values decrease along any p-path.

$$f(\sigma_{j_i}) \geq f(\tau_{j_i}) > f(\sigma_{j_{i+1}}), \quad \forall i.$$

In particular: $f(\sigma_{i_1}) > f(\sigma_{i_m})$, $\forall m > 1$.

ⓒ ⓑ implies each DGVF is acyclic.

↙ converse



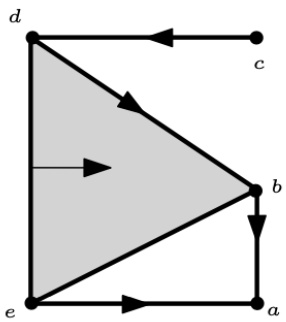


**THM:** Each acyclic DVF on K is a GDVF.

(Related to: a vector field on $\mathbb{R}^2$ with zero curl is a grad. field)

**Proof by example:**

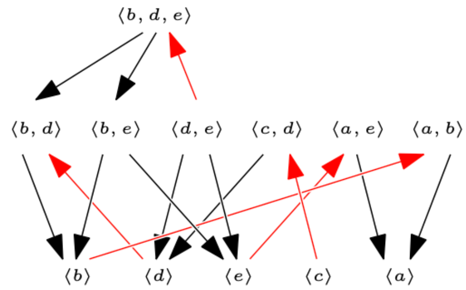
Acyclic DVF



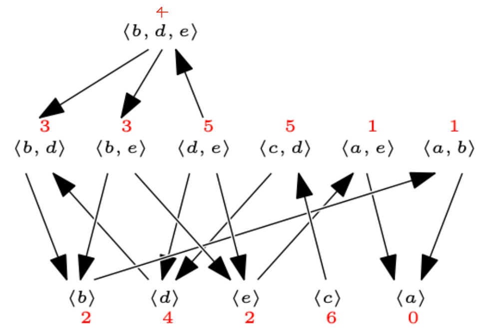


Modified Hasse diagram encoding conditions on f.

acyclic graph

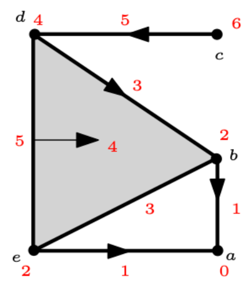








reverse engineer values of f.





■

**Prop:** Suppose critical sxes of an acyclic DVF on K form a subcx $L \leq K$. Then $K \searrow L$ and thus $K \simeq L$.

**Cor:** If an acyclic DVF on K has a single critical sx, then $K \simeq \bullet$.

**Proof:** Claim: $\exists$ a regular pair $(\sigma, \tau)$ with $\sigma$ being a free face.

↙ justification

Let n --- max dim of a sx in $K \setminus L$.
Choose a maximal n-path.
Let $(\sigma, \tau)$ be its initial pair

$\sigma$ is a free face:

- $\sigma < \tau$ by def
- If $\sigma$ was a facet of a sx in $K \setminus L$, the pair of that sx could be used to prolong our p-path. →←
- If $\sigma$ was a facet of a sx in L, $\sigma$ itself would be in L →← □

how to use it

- Remove the pair $(\sigma, \tau)$ by an elementary collapse
- Inductively use the claim to proceed
- end @ L.

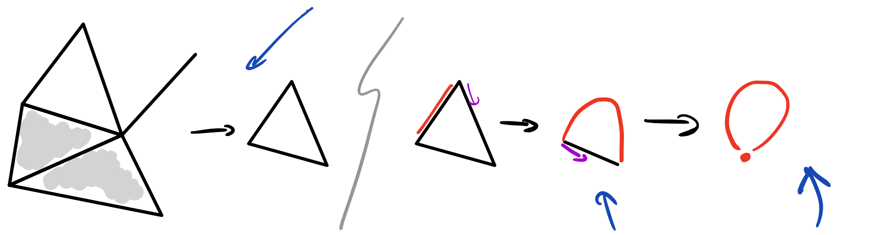
■

# ④ Morse homology (How to compute homology from critical sxes).

discrete ← setting

**Classical Morse theory:** Obtain homology or a homotopy type of a manifold from critical points of a function on it. (on $S^1$: # of MAX = # of MIN)
**Similarly:** a manifold admits a non-trivial tangent vector field $\Rightarrow \chi = 0$.

IDEA: last time we simplified.



Today we choose a crit sx and continue to further simplification (not a sx).

SETTING: $K$ a scx with a gradient vector field, $G$ a group for coefficients

$n_i$ ... # of critical $i$-sxes

**Def:** let $p \in \{0,1,..\}$. A **Morse $p$-chain** is a formal sum

$$\sum_{i=1}^{n_p} \lambda_i \cdot \sigma_i^p \qquad \lambda_i \in G,\ \sigma_i^p \in K \text{ oriented critical } p\text{-sx}$$

**Morse chain group** $C_p$ is the group of Morse $p$-chains (with the obvious operations).

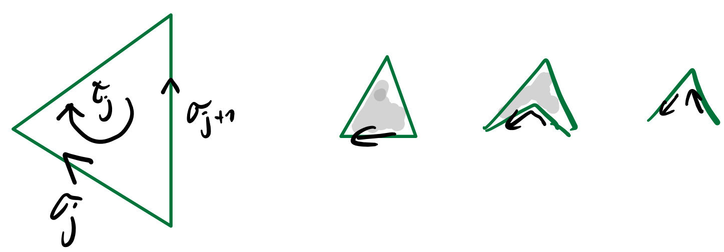
An **oriented $p$-path** from an oriented sx $\sigma_1^{p-1}$ to an oriented sx $\sigma_{k+1}^{p-1}$ is a $p$-path

$$\sigma_1^{p-1} \to \tau_1^p \geq \sigma_2^{p-1} \to \tau_2^p \geq \sigma_3^{p-1} \to \dots \to \tau_k^p \geq \sigma_{k+1}^{p-1}$$

consisting of oriented sxes, such that for each $j$ the orientation induced by $\tau_j$ on its faces:

① Matches $\sigma_j$

② Does not match $\sigma_{j+1}$




Given oriented sx $\tau^p$ let $\delta(\tau)$ denote the collection of all of its facets with the induced orientation arising from $\tau$.

For each oriented critical $(p-1)$-sx $\sigma$ let

$$d_{\tau,\sigma} = \sum_{\sigma' \in \delta(\tau)} |\{\text{oriented } p\text{-paths from } \sigma' \text{ to } \sigma\}|$$

**Def:** The boundary map $\partial$ on $C_p$ is defined as follows (on the generators $\in$ crit. p-sxes)

$$\partial_p \tau = \sum_{i=1}^{n_{p-1}} \left(\underbrace{\alpha_{\tau,\sigma_i}} - \underbrace{\alpha_{\tau,-\sigma_i}}\right)\cdot \sigma_i$$

in general not related

## Morse chain complex:

$$\cdots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

It turns out $\partial^2 = 0$.

## Morse homology

$$\mathcal{H}_p(K;G) = \ker\partial_p \big/ \operatorname{Im}\partial_{p+1}$$

quotient also depends on grad. vect. field, but not the homology

**THM:** Morse homology is isomorphic to simplicial homology:

$$\mathcal{H}_p(K;\mathbb{F}) \cong H_p(K,\mathbb{F}).$$

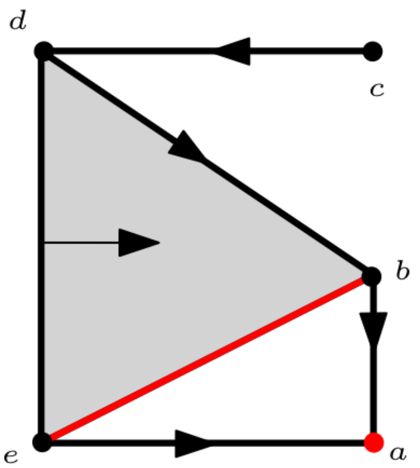
**Corollary:** $\forall p \quad n_p \geq b_b$ ← Betti number

\# of crit p-sxes

⟶ If for some DMF we have $n_p = b_p, \forall p$, $f$ is called perfect (and $\partial_p = 0, \forall p$ if $G$ is a field). We usually strive to get it but not all spaces admit it. [Example: Dunce hat ≃ but no free face].

**Examples:** (1)

generate DMF first



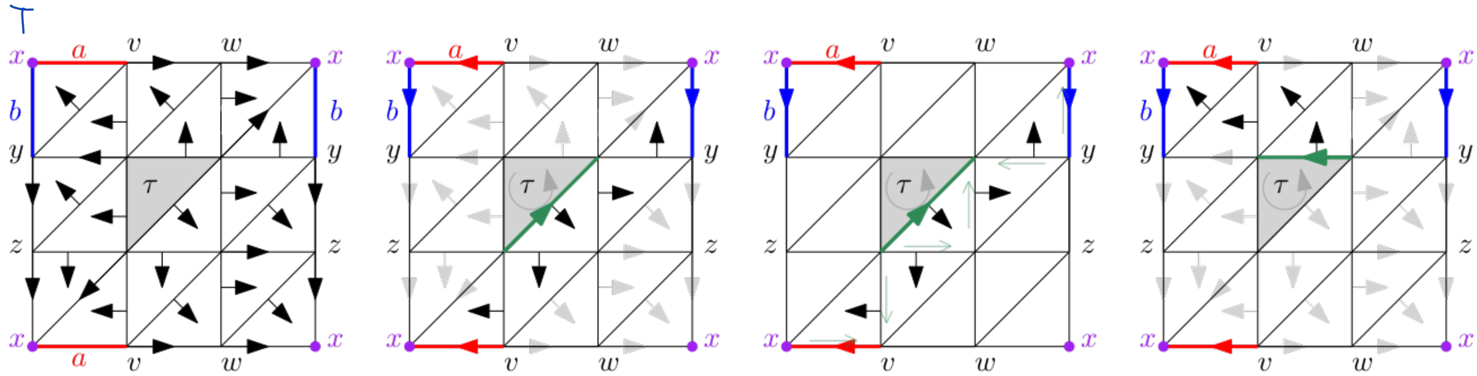

chain cx:

$$0 \longrightarrow \underset{\overset{\shortparallel}{C_1}}{\mathbb{R}\langle b,e\rangle} \longrightarrow \underset{\overset{\shortparallel}{C_0}}{\mathbb{R}\langle a\rangle} \longrightarrow 0$$

$\partial\langle b,e\rangle = \text{endpt}(e\to a) - \text{endpt}(b - a) = a - a = 0$

$$\boxed{\begin{matrix} b_1 = 1 \\ b_0 = 1 \end{matrix}}$$

(2) T




$$\partial\tau = \underbrace{-\langle b\rangle - \langle a\rangle \qquad\qquad +\langle b\rangle + \langle a\rangle}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2 & \xrightarrow{0} & C_1 & \xrightarrow{0} & C_0 & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{R}_\tau & & \mathbb{R}^2_{\langle a\rangle, \langle b\rangle} & & \mathbb{R}_{\langle x\rangle} & & \end{array}$$

$$\left.\begin{array}{l} b_2 = 1 \\ b_1 = 2 \\ b_0 = 1 \end{array}\right\} \text{Only requires 4 crit. sxs!!}$$

③ generating DMF on graphs $\xrightarrow{\text{via spanning tree}}$