w interleaving

Stability of Persistence

① Cont's filtrations, + 10 mins
② P. modules up to interleaving, +5 mins
③ d_H, d_{GH}

Next: Bottleneck distance.

① Continuous filtrations

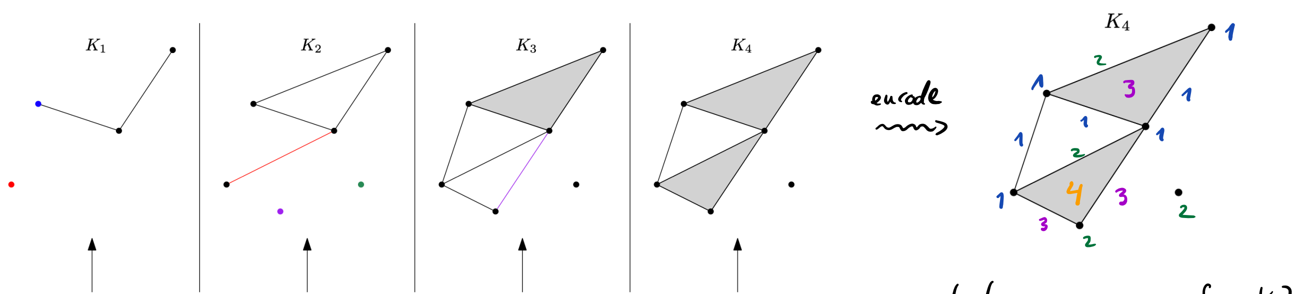
Def: A continuous filtration of a scx K is a collection of subcomplexes $\{K_r\}_{r\geq 0}$ of K:

$$\forall r<q: \; K_r \subseteq K_q.$$

! To make life easier we assume: $\forall \sigma \in K$ $\underset{r}{\arg\min}\, \{\sigma \in K_r\}$ exists.

σ appears AT some scale, not Beyond some scale.
("closed" notation @ defs of cxes)

Examples: Rips, Čech, Alpha, ...



discrete filtration

annotation $f(\sigma) = \underset{r}{\arg\min}\{\sigma \in K_r\}$

Def: Filtration function for scx K: map $K \to [0,\infty)$, $\sigma \mapsto f(\sigma)$ such that $\tau \subseteq \sigma \Rightarrow f(\tau) \leq f(\sigma)$.

Sublevel filtration of f: $\{ f^{-1}[0,r]\}_{r\geq 0}$ ⟸ continuous filtration

$$\text{continuous filtration} \underset{\text{sublevel filtr.}}{\overset{\text{annotation}}{\rightleftarrows}} \text{filtration function}$$

Example: Filtration functions: Rips ⇝ diam

Čech ⇝ radius of smallest enclosing ball.

! Computation of Persistence is unhindered as it only requires the finitely many annotation values.

Example: [offsets] $S \subset \mathbb{R}^n$ finite

$\{ \check{C}ech\,(S,r)\}_{r\geq 0} \underset{\text{Nerve THM}}{\simeq} \{ N(S,r)\}_{r\geq 0}$ models growth of $\overbrace{\text{neighborhoods}}^{\text{offsets}}$. The changes in homology can be computed & tracked by pers. hom.

Def: $\varepsilon \geq 0$. Filtrations $\{K_r\}_{r\geq 0}$ and $\{L_r\}_{r\geq 0}$ are **ε-interleaved** if

$\exists$ simplicial maps $\varphi_r : K_r \to L_{r+\varepsilon}$ $\quad \psi_r : L_r \to K_{r+\varepsilon}$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & K_r \arrow[rr] \arrow[drr, "\varphi_r"' near end] & & K_{r+\varepsilon} \arrow[rr] \arrow[drr] & & K_{r+2\varepsilon} \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & L_r \arrow[rr] \arrow[urr, "\psi_r" near end] & & L_{r+\varepsilon} \arrow[rr] \arrow[urr] & & L_{r+2\varepsilon} \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

Isomorphism = 0-interleaving.

Interleaving distance: $\arg\inf_{\varepsilon} \left\{ \{K_r\}_{r\geq 0} \text{ and } \{L_r\}_{r\geq 0} \ \varepsilon\text{-interleaved} \right\}$ $\leftarrow$ actually arg min in our case

MAX-distance between filtration functions f, g on K: $\quad$ $\uparrow$ metric on equiv. classes of filtrations

$$\|f-g\|_\infty = \max_{\sigma \in K} |f(\sigma) - g(\sigma)|$$

Proposition: Sublevel filtrations of f and g are $\|f-g\|_\infty$-interleaved. $\leftarrow$ for example cubical cx of images

Proof: ψ_r & φ_r are identities on vertices.

observe: $\forall \sigma \in K$: $\quad f(\sigma) \leq r \Rightarrow g(\sigma) \leq r+\varepsilon$ $\quad$ & vice versa

This concludes the proof. $\blacksquare$

Also recall: $X = \{x_1, \dots, x_k\}$, $Y = \{y_1, \dots, y_k\}$, $\quad d(x_i, y_i) \leq \varepsilon$. Then:

1. Rips filtrations of X & Y are 2ε-interleaved
2. Čech filtrations of X & Y are ε-interleaved.

---

2 Persistence modules: fix $\mathbb{F}$ $\leftarrow$ obtained from filtrations by applying homology

Def: A **persistence module** is a collection of finite-dimensional vector spaces $\{V_r\}_{r\geq 0}$

along with linear maps $\quad h_{r,q} : V_r \to V_q, \quad \forall r \leq q$, such that

$h_{r,r} = id_{V_r}$ $\quad$ and $\quad$ $h_{q,s} \circ h_{r,q} = h_{r,s}$ $\quad \forall r \leq q \leq s$.

Scale r is **regular** if $\exists \varepsilon > 0$; $h_{q,s}$ is an isomorphism, $\forall q \leq s \in (r-\varepsilon, r+\varepsilon)$.

Scale r is **critical** if it is not regular.

Additional assumptions:
- Eventually all $h_{*,*}$ are isomorphisms: $\exists R$: $\forall q \geq r \geq R$ $\quad h_{r,q}$ is isomorphism
- $\forall r \geq 0 \ \exists r' > 0$; $\forall q \in [r, r')$ $\quad h_{q,r'}$ is isomorphism
- There are only finitely many critical scales

$\Big\}$ hold for homology of our filtrations

Def: $\varepsilon > 0$. Persis. modules $\{V_r\}_{r\geq 0}$ and $\{W_r\}_{r\geq 0}$ are **ε-interleaved** if

$\exists$ linear maps $\varphi_r : V_r \to W_{r+\varepsilon}$ $\quad \psi_r : W_r \to V_{r+\varepsilon}$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & V_r \arrow[rr] \arrow[drr] & & V_{r+\varepsilon} \arrow[rr] \arrow[drr] & & V_{r+2\varepsilon} \arrow[r] & \cdots \\
\cdots \arrow[r] & W_r \arrow[rr, "\varphi_r"'] \arrow[urr, "\psi_r"] & & W_{r+\varepsilon} \arrow[rr] \arrow[urr] & & W_{r+2\varepsilon} \arrow[r] & \cdots
\end{tikzcd}$$

Def: Persistence modules $\{V_r\}_{r\geq 0}$ and $\{W_r\}_{r\geq 0}$ are **isomorphic**, if they are 0-interleaved:

$\exists$ isomorphisms $V_r \xrightarrow{\cong} W_r$ commuting with the maps h:
$$\begin{tikzcd}
\cdots \arrow[r] & V_r \arrow[r, "h^V"] \arrow[d, "\cong"'] & V_s \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & W_r \arrow[r, "h^W"] & W_s \arrow[r] & \cdots
\end{tikzcd}$$

Interleaving distance: $\arg\inf_{\varepsilon} \Big\{ \{V_r\}_{r\geq 0} \text{ and } \{W_r\}_{r\geq 0} \ \varepsilon\text{-interleaved} \Big\}$ ← actually arg min in our case

↑ metric on isom. classes of pers. modules

Observations: ε-interleaved filtrations $\xrightarrow[H_q]{\text{induce}}$ ε-interleaved p. modules.

Barcodes are "horizontal decomposition" of pers. modules. Let us formalize it:

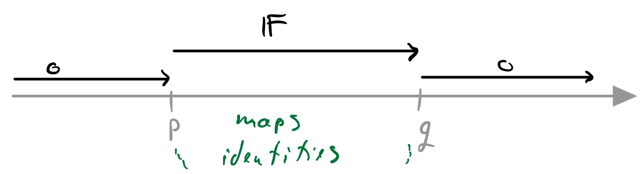
Def: The **direct sum** of persistence modules $\{V_r\}_{r\geq 0}$ and $\{W_r\}_{r\geq 0}$ is a persist. module $\{V_r \oplus W_r\}_{r\geq 0}$ with bonding maps $h^V \oplus h^W$.

Def: $0 \leq p < q$. **Interval module** $\mathbb{F}_{p,q}$ is a persist. module $\{V_r\}_{r\geq 0}$ with

$V_r = \mathbb{F}$ if $r \in [p,q)$

$V_r = 0$ if $r \notin [p,q)$

bonding maps are isomorphisms whenever possible



Theorem: [Structure theorem for persistent homology] Each persistence module is isomorphic to a (finite) direct sum of interval modules. The decomposition is unique up to permutation of the intervals.

$\boxed{\text{Barcode decomposition formally}}$

Example: Interleaving distance between interval modules $= d_B$ between PD's.

Are there interleavings on p. modules not arising from interleav. of filtrations? Yes! By "similar spaces". Let's explain it.

Def: X a metric space, $A, B \subseteq X$ finite subspaces.

Hausdorff distance $\quad d_X(A,B) = \max\left\{\max_{a\in A}\min_{b\in B} d(a,b),\ \max_{b\in B}\min_{a\in A} d(a,b)\right\}$

is a metric on finite subsets of X.

Def: A, B finite metric spaces.

Gromov-Hausdorff distance $\quad d_{GH}(A,B) = \inf_{X,\nu,\mu} d_H(\nu(A), \mu(B))$

$\nu: A \hookrightarrow X$, $\mu: B \hookrightarrow X$ isometric embeddings.

is a metric on isometry classes of finite metric spaces.

inf always attained.

Proposition: $d_{GH} \leq d_H$

Proposition: $\varepsilon > 0$; A, B finite metric spaces, $d_{GH}(A,B) \leq \varepsilon$, $q \in \{0, 1, \dots\}$. Then:

ⓐ $\{H_q(\mathrm{Rips}(A,r);\mathbb{F})\}_{r\geq 0}$ and $\{H_q(\mathrm{Rips}(B,r);\mathbb{F})\}_{r\geq 0}$ are 2ε-interleaved.

ⓑ $\{H_q(\check{C}\mathrm{ech}(A,r);\mathbb{F})\}_{r\geq 0}$ and $\{H_q(\check{C}\mathrm{ech}(B,r);\mathbb{F})\}_{r\geq 0}$ are ε-interleaved.

Proof: [for Rips, $q=1$]

$$\begin{tikzcd}
H_1(\mathrm{Rips}(A,r);\mathbb{F}) \arrow[r] \arrow[dr, "\varphi_r"] & H_1(\mathrm{Rips}(B,r+2\varepsilon);\mathbb{F}) \\
H_1(\mathrm{Rips}(B,r);\mathbb{F}) \arrow[r] \arrow[ur, "\psi_r"] & H_1(\mathrm{Rips}(A,r+2\varepsilon);\mathbb{F})
\end{tikzcd}$$

Assume $A, B \subset X$ by def.

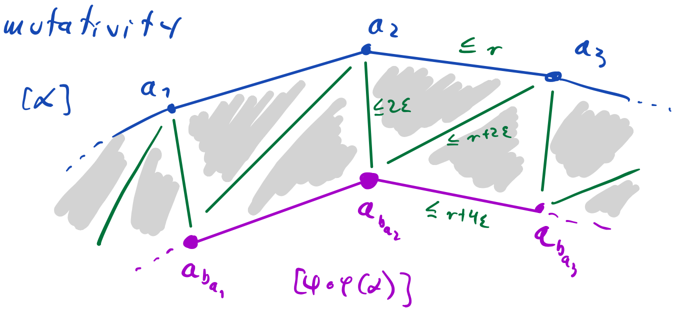
$\forall a \in A$ choose $b_a \in B$: $d(a, b_a) \leq \varepsilon$. $\forall b \in B$ choose $a_b \in A$: $d(a,b) \leq \varepsilon$

Take cycle $\alpha = \langle a_1, a_2\rangle + \langle a_2, a_3\rangle + \dots + \langle a_k, a_1\rangle$

Def $\quad \varphi_r([\alpha]) = [\langle b_{a_1}, b_{a_2}\rangle + \dots + \langle b_{a_k}, b_{a_1}\rangle]$. $\Leftarrow$ turns out to be a well-defined lin. map.

$\quad \psi_r([\langle b_1, b_2\rangle + \dots + \langle b_m, b_1\rangle]) = [\langle a_{b_1}, a_{b_2}\rangle + \dots + \langle a_{b_m}, a_{b_1}\rangle]$

Commutativity



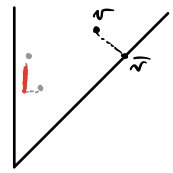
all sides $\leq r + 2\varepsilon$

$\Leftarrow$ homology between $[\alpha]$ & $\psi\circ\varphi([\alpha])$

③ Bottleneck distance: a metric on persistence diagrams.

For $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$ define: $d_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|, |y_1-y_2|\}$

$$\overline{(x_1,y_1)} = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$$



Assume $\mathcal{A}=(a_1,a_2,\dots,a_m)$ and $\mathcal{B}=(b_1,b_2,\dots,b_n)$ are persistence diagrams.

a_i, b_i pts in $\{(x,y)\in\mathbb{R}^2; y>x\ge 0\}$, appearing possibly with repetitions in $\mathcal{A}$ & $\mathcal{B}$.

Def: A partial matching between $\mathcal{A}$ and $\mathcal{B}$ is a bijective map

$\varphi: \mathcal{A}' \to \mathcal{B}'$ for some $\mathcal{A}'\subseteq\mathcal{A}$, $\mathcal{B}'\subseteq\mathcal{B}$.

The matching distance of φ is

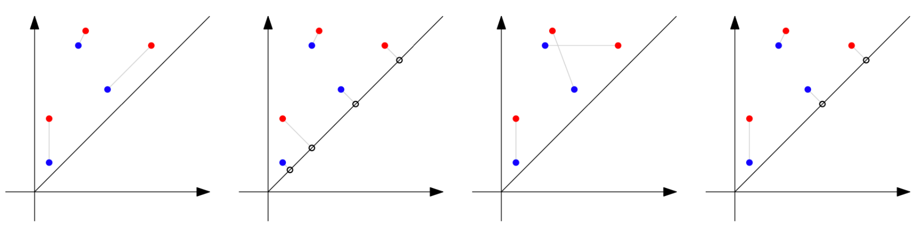
$$d_M(\varphi)=\max\left\{\max_{v\in\mathcal{A}'}\{d_\infty(v,\varphi(v))\}, \max_{v\in\mathcal{A}\setminus\mathcal{A}'}\{d_\infty(v,\bar v)\}, \max_{v\in\mathcal{B}\setminus\mathcal{B}'}\{d_\infty(v,\bar v)\}\right\}$$

Let $\mu(\mathcal{A},\mathcal{B})$ be the collection of partial matchings $\mathcal{A}$ to $\mathcal{B}$.

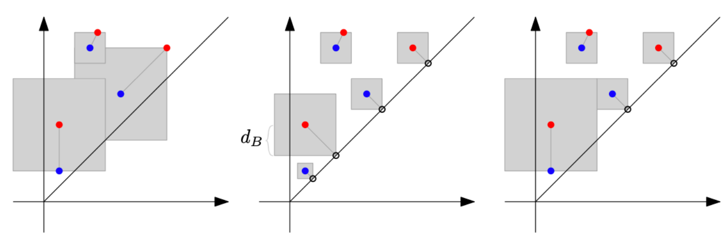
The Bottleneck distance is defined as

$$d_B(\mathcal{A},\mathcal{B}) = \min_{\varphi\in\mu(\mathcal{A},\mathcal{B})} d_M(\varphi)$$

partial matchings



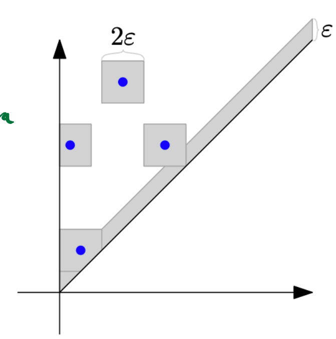
Bottleneck distance



Isometry Theorem: The interleaving distance between persistence modules } algebra

EQUALS

the bottleneck distance between persistence diagrams. visualisation



Stability Theorem:

$$\begin{tikzcd}
\boxed{\begin{array}{l}\varepsilon-\textit{interleaving of}\\ \textit{filtrations}\\ \mathcal{V} \textit{ and } \mathcal{W}\end{array}} \arrow[rr, "\textit{functoriality}"] & & \boxed{\begin{array}{l}\varepsilon-\textit{interleaving of}\\ \textit{persistence}\\ \textit{modules}\end{array}} \arrow[r, "\textit{Isometry}", "\textit{Theorem}"'] & \boxed{d_B(\mathcal{A},\mathcal{B}) \leq \varepsilon} \\
\boxed{\begin{array}{l}\textit{Filtration}\\ \textit{functions}\\ \|f-g\|_\infty \leq \varepsilon\end{array}} \arrow[u] & \boxed{\begin{array}{l}\textit{Rips}\\ \textit{filtration}\\ d_{GH}(X,Y) \leq \varepsilon/2\end{array}} \arrow[ur] & & \boxed{\begin{array}{l}\textit{Cech}\\ \textit{filtration}\\ d_{GH}(X,Y) \leq \varepsilon\end{array}} \arrow[ul]
\end{tikzcd}$$

Alternative metric (sensitive to more changes):

Wasserstein distance:

let $p > 1$

The p-matching distance of φ is

$$d_M^p(\varphi) = \sum_{v\in\mathcal{A}'} \{ d_\infty(v,\varphi(v)) + \sum_{v\in\mathcal{A}\setminus\mathcal{A}'} \{ d_\infty(v,\bar{v}) + \sum_{v\in\mathcal{B}\setminus\mathcal{B}'} d_\infty(v,\bar{v})$$

The p-Wasserstein distance is defined as

$$W_p(\mathcal{A},\mathcal{B}) = \min_{\varphi\in\mu(\mathcal{A},\mathcal{B})} d_M(\varphi)$$

W_p is also stable under certain conditions.

!!crirical sxes not stable!!