Ime in priimek

Vpisna številka

1	
2	
3	
4	
Σ	

Linearna algebra: računski izpit

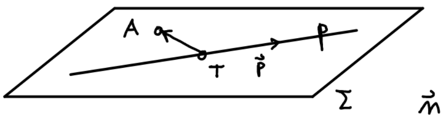
4. september 2024

Čas pisanja: 90 minut. Dovoljena je uporaba dveh listov velikosti A4 z obrazci. Uporaba elektronskih pripomočkov ni dovoljena. Rezultati bodo objavljeni na *ucilnica.fri.uni-lj.si*. **Vse odgovore dobro utemelji!**

1. naloga (25 točk)

Dani sta točka $A(0,-2,3)$ in premica $p : \frac{x-3}{2} = y+1 = z-2$.

a) (5) Določi enačbo ravnine Σ, ki vsebuje točko A in premico p.

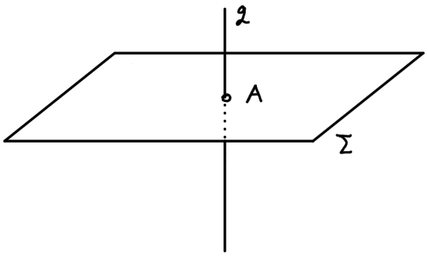


$\vec p = \begin{bmatrix}2\\1\\1\end{bmatrix}$ $\quad T(3,-1,2)$ $\quad \vec{TA} = \begin{bmatrix}-3\\-1\\1\end{bmatrix}$

$\vec n \parallel \vec{TA}\times\vec p = \begin{bmatrix}-3\\-1\\1\end{bmatrix}\times\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\5\\-1\end{bmatrix}$ $\quad \vec n = \begin{bmatrix}2\\-5\\1\end{bmatrix}$ $\quad \vec n\cdot\vec r_A = 13$

$\Sigma:\ 2x-5y+z=13$

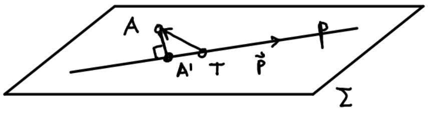
b) (5) Določi enačbo premice q, ki je pravokotna na ravnino Σ in vsebuje točko A.



$q\perp\Sigma \rightarrow \vec q\parallel\vec n$

$q:\ \begin{bmatrix}0\\-2\\3\end{bmatrix} + t\begin{bmatrix}2\\-5\\1\end{bmatrix}$

c) (10) Določi projekcijo A' točke A na premico p.



$p:\ \begin{bmatrix}3\\-1\\2\end{bmatrix} + s\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}3+2s\\-1+s\\2+s\end{bmatrix}$

$\vec{AA'} = \vec r_{A'} - \vec r_A = \begin{bmatrix}3+2s\\-1+s\\2+s\end{bmatrix} - \begin{bmatrix}0\\-2\\3\end{bmatrix} = \begin{bmatrix}3+2s\\1+s\\-1+s\end{bmatrix} \perp \vec p$

$\vec{AA'}\cdot\vec p = 2(3+2s)+1(1+s)+1(-1+s) = 6+4s+1+s-1+s = 0$

$6s+6=0$

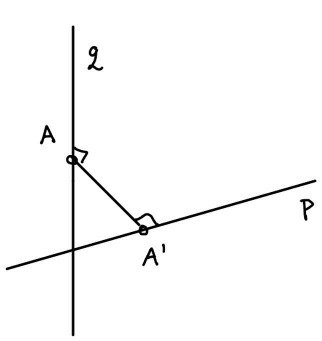
$6s=-6$

$s=-1$

$\begin{bmatrix}1\\-2\\1\end{bmatrix}$

$A'(1,-2,1)$

d) (5) Določi razdaljo med premicama p in q.



$\left.\begin{matrix}\vec{AA'}\subseteq\Sigma \rightarrow \vec{AA'}\perp\vec q\\ \vec{AA'}\perp\vec p\end{matrix}\right\}$ $d(p,q) = \|\vec{AA'}\| = \left\|\begin{bmatrix}1\\0\\-2\end{bmatrix}\right\| = \sqrt{1+4} = \sqrt5$

2. naloga (25 točk)

Naj bo

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{in} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ \alpha \end{bmatrix},$$

kjer je $\alpha \in \mathbb{R}$.

a) (10) Poišči vse rešitve sistema $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 2 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & \alpha \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 4 & 1 & 2 \\ 0 & -4 & 3 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 1 & 1 & \alpha \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 4 & 1 & 2 \\ 0 & 1 & 1 & \alpha \\ 0 & -2 & 5 & -2 \\ 0 & -4 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -3 & 2-4\alpha \\ 0 & 1 & 1 & \alpha \\ 0 & 0 & 7 & -2+2\alpha \\ 0 & 0 & 7 & 4\alpha \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & -3 & 2-4\alpha \\ 0 & 1 & 1 & \alpha \\ 0 & 0 & 1 & \frac{2\alpha-2}{7} \\ 0 & 0 & 1 & \frac{4\alpha}{7} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-11\alpha+4}{7} \\ 0 & 1 & 0 & \frac{5\alpha+2}{7} \\ 0 & 0 & 1 & \frac{2\alpha-2}{7} \\ 0 & 0 & 0 & \frac{2\alpha+2}{7} \end{array}\right]$$

poln rang → edina rešitev $A\vec{x} = \vec{0}$ je $\underline{\underline{\vec{x} = \vec{0}}}$

b) (10) Ali obstaja tak $\alpha \in \mathbb{R}$, da bo imel sistem $A\vec{x} = \vec{b}$ več kot eno rešitev?

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 2 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & \alpha \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-11\alpha+4}{7} \\ 0 & 1 & 0 & \frac{5\alpha+2}{7} \\ 0 & 0 & 1 & \frac{2\alpha-2}{7} \\ 0 & 0 & 0 & \frac{2\alpha+2}{7} \end{array}\right]$$

Sistem bo rešljiv le za $\frac{2\alpha+2}{7} = 0$, torej za $\alpha = -1$. V tem primeru bo rešitev enolična: $x_1 = \frac{15}{7}$, $x_2 = -\frac{3}{7}$, $x_3 = -\frac{4}{7}$ oz. $\vec{x} = \begin{bmatrix} 15/7 \\ -3/7 \\ -4/7 \end{bmatrix}$. Za noben α torej nima neskončno rešitev.

c) (5) Ali obstaja tak $\alpha \in \mathbb{R}$, da bo sistem $A\vec{x} = \vec{b}$ protisloven?

Ja, za vse $\alpha \neq -1$ je protisloven.

3. naloga (25 točk)

Naj bo

$$A = \begin{array}{c} \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \\ \begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & -4 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}.$$

a) (10) Poišči *ortonormirano bazo* stolpčnega prostora matrike A.

$$\vec{u}_1 = \vec{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \vec{a}_2 - \frac{\vec{a}_2 \vec{u}_1}{\vec{u}_1 \vec{u}_1} \vec{u}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{u}_3 = \vec{0}$, ker je $\vec{a}_3 = 2\vec{a}_2 - \vec{a}_1 \in \mathcal{L}in\{\vec{a}_1, \vec{a}_2\} = \mathcal{L}in\{\vec{u}_1, \vec{u}_2\}$

$$\|\vec{u}_1\| = \|\vec{u}_2\| = \sqrt{4+4+1} = \sqrt{9} = 3 \qquad \vec{q}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \\ 0 \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} 2/3 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$\{\vec{q}_1, \vec{q}_2\}$ je ONB za $C(A)$

b) (10) Poišči pravokotno projekcijo vektorja $\vec{v} = [-1, 5, 1, 3]^{\mathsf{T}}$ na $C(A)$.

$$\mathrm{proj}_{C(A)}(\vec{v}) = (\vec{v} \cdot \vec{q}_1)\,\vec{q}_1 + (\vec{v} \cdot \vec{q}_2)\,\vec{q}_2 = \left(-\frac{2}{3} + \frac{10}{3} + \frac{1}{3} + 0\right)\vec{q}_1 + \left(-\frac{2}{3} - \frac{10}{3} + 0 + \frac{3}{3}\right)\vec{q}_2 =$$

$$= 3\vec{q}_1 - 3\vec{q}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix}}}$$

c) (5) Poišči pravokotno projekcijo vektorja $\vec{v} = [-1, 5, 1, 3]^{\mathsf{T}}$ na $C(A)^{\perp}$.

$$\mathrm{proj}_{C(A)^{\perp}}(\vec{v}) = \vec{v} - \mathrm{proj}_{C(A)}(\vec{v}) = \begin{bmatrix} -1 \\ 5 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \end{bmatrix}}}$$

4. naloga (25 točk)

Zaporedji a_n in b_n sta podani rekurzivno z začetnima členoma $a_0 = 2$, $b_0 = 0$ in zvezama

$$\begin{aligned} a_n &= a_{n-1} + 3b_{n-1}, \\ b_n &= 3a_{n-1} + b_{n-1}. \end{aligned}$$

a) (5) Poišči matriko A, za katero lahko sistem rekurzivnih formul zapišemo v matrični obliki kot

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = A \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}.$$

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \underline{\underline{\overset{\overset{A}{\shortparallel}}{\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}}}} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}$$

b) (10) Poišči lastni vrednosti $\lambda_1 > \lambda_2$ in pripadajoča lastna vektorja $\vec{v}_1$ in $\vec{v}_2$ matrike A.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda + 1 - 9 = \lambda^2 - 2\lambda - 8 = (\lambda+2)(\lambda-4)$$

$$\underline{\underline{\lambda_2 = -2}} \qquad \underline{\underline{\lambda_1 = 4}}$$

- $A - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = x_2 \qquad \underline{\underline{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}}$

- $A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = -x_2 \qquad \underline{\underline{\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}}$

c) (5) Začetni vektor $\vec{x}_0 = [a_0, b_0]^{\mathsf{T}} = [2, 0]^{\mathsf{T}}$ razvij po bazi $\{\vec{v}_1, \vec{v}_2\}$.

$$\underline{\underline{\vec{x}_0}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underset{1}{\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underset{-1}{\beta} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underline{\underline{\vec{v}_1 - \vec{v}_2}}$$

d) (5) Za $\vec{x}_n = [a_n, b_n]^{\mathsf{T}}$ iz zveze $\vec{x}_n = A^n \vec{x}_0$ izpelji splošni formuli za a_n in b_n.

$$\vec{x}_n = A\vec{x}_{n-1} = A^2\vec{x}_{n-2} = \cdots A^{n-1}\vec{x}_1 = A^n\vec{x}_0 = A^n(\vec{v}_1 - \vec{v}_2) = A^n\vec{v}_1 - A^n\vec{v}_2 =$$

$$= 4^n\vec{v}_1 - (-2)^n\vec{v}_2 = \begin{bmatrix} 4^n \\ 4^n \end{bmatrix} - \begin{bmatrix} -(-2)^n \\ (-2)^n \end{bmatrix} = \begin{bmatrix} 4^n + (-2)^n \\ 4^n - (-2)^n \end{bmatrix}$$

$$\underline{\underline{a_n = 4^n + (-2)^n}}$$

$$\underline{\underline{b_n = 4^n - (-2)^n}}$$