Ime in priimek

Vpisna številka

1	
2	
3	
Σ	

Linearna algebra: 2. računski izpit

28. junij 2021

Čas pisanja: 70 minut. Dovoljena je uporaba dveh listov velikosti A4 z obrazci. Uporaba elektronskih pripomočkov ni dovoljena. Rezultati bodo objavljeni na *ucilnica.fri.uni-lj.si*. **Vse odgovore dobro utemelji!**

1. naloga (15 točk)

Dani sta premica p in ravnina Σ:

$$p \;:\; \frac{x+2}{3} = \frac{y}{2} = z-2,$$
$$\Sigma \;:\; x+3y-2z=1.$$

a) (5) Izračunaj kot med smernim vektorjem premice p in normalo ravnine Σ.

$\vec{p} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ 1 $\qquad \vec{n} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ 1

$\vec{p}\cdot\vec{n} = \|\vec{p}\|\cdot\|\vec{n}\|\cdot\cos\varphi$ 1

$3+6-2 = \sqrt{14}\,\sqrt{14}\,\cos\varphi$

$7 = 14\cos\varphi$

$\cos\varphi = \frac{1}{2}$

$\varphi = \frac{\pi}{3}$ $(60°)$ 2

b) (5) Izračunaj presek premice p in ravnine Σ.

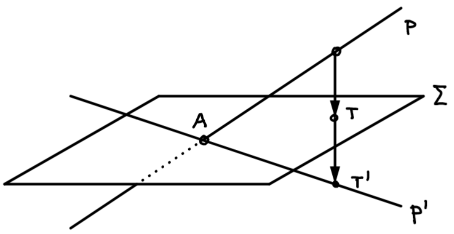
$\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+3t \\ 2t \\ 2+t \end{bmatrix} \in \Sigma \Rightarrow -2+3t+6t-4-2t=1$ 2

$7t = 7$

$t=1$

$A(1,2,3) \in \Sigma \cap p$ 3

c) (5) Poišči enačbo premice p', ki jo dobimo z zrcaljenjem premice p čez ravnino Σ.



$\vec{r}_T = \vec{r}_P + \lambda\vec{n} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2+\lambda \\ 3\lambda \\ 2-2\lambda \end{bmatrix} \in \Sigma$

$-2+\lambda+9\lambda-4+4\lambda=1$

$14\lambda = 7$

$\lambda = \frac{1}{2}$

$\vec{r}_{T'} = \vec{r}_P + 2\lambda\vec{n} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + 2\cdot\frac{1}{2}\cdot\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ 2

$\vec{p}\,' = \overrightarrow{AT'} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ 2

$p' : \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ 1

2. naloga (15 točk)

Dani so matrika A ter vektorja $\vec{b}_1$ in $\vec{b}_2$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}.$$

a) (5) Kateri od sistemov $A\vec{x} = \vec{b}_1$ in $A\vec{x} = \vec{b}_2$ je enolično rešljiv? Kaj je rešitev?

$$\begin{array}{c} \begin{array}{ccccc} & A & & \vec{b}_1 & \vec{b}_2 \end{array} \\ \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 1 & 3 & 1 & -1 \end{array}\right] \end{array} \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 1 & 3 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & 0 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 15 \\ 0 & 0 & 1 & 0 & -5 \end{array}\right]$$

2 1

$A\vec{x} = \vec{b}_2$ je protisloven

Sistem $A\vec{x} = \vec{b}_1$ je enolično rešljiv, rešitev je $\vec{x} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$. 2

b) (10) Za sistem $A\vec{x} = \vec{b}_2$ poišči približek za rešitev po linearni metodi najmanjših kvadratov.

$$A^TA = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 5 & 13 \end{bmatrix}$$

3

$$A^T\vec{b}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

3

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 1 & 3 & 5 & 4 \\ 0 & 5 & 13 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & 5 & 10 & 5 \\ 0 & 5 & 13 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

2

Rešitev po MNK je $\vec{x} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$.

2

3. naloga (20 točk)

Dani sta matriki
$$J = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ in } N = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

a) (5) Prepričaj se, da imata matriki J in $J+N$ enaka karakteristična polinoma (in zato enake lastne vrednosti).

$$\det(J-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -(\lambda+1)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -(\lambda+1)\underbrace{\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}}_{(1-\lambda)^2-4=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)} = -(\lambda+1)^2(\lambda-3) \quad ②$$

$$\det(J+N-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ -1 & -1-\lambda & 1 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -(\lambda+1)\begin{vmatrix} 1-\lambda & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -(\lambda+1)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-3) \quad ③$$

b) (5) Poišči lastne vrednosti matrik J in $J+N$.

$\lambda_{1,2} = -1, \ \lambda_3 = 3$ ⑤

c) (5) Diagonaliziraj matriko J v ortonormirani bazi $\mathbb{R}^3$ – poišči diagonalno matriko D ter ortogonalno matriko Q, da bo $J = QDQ^\mathsf{T}$.

$$J+I = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to x_1 = -x_3 \qquad \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ ②$$

$$J-3I = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to x_1 = x_3 \\ \to x_2 = 0 \end{matrix} \qquad \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} \to \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \ ①$$

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \ ① \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \ ①$$

d) (5) Se da matriko $J+N$ tudi diagonalizirati? Zakaj oz. zakaj ne?

$$J+N+I = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to x_1 = 0 \\ \to x_3 = 0 \end{matrix} \qquad \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ ②$$

Ker je $\dim N(J+N+I) = 1$ in algebraična kratnost l. vrednosti $\lambda_{1,2} = -1$ enaka 2, matrika $J+N$ ni diagonalizabilna. ③