Mathematical Modelling Exam

May 27th, 2024

You have 75 minutes to solve the problems. The numbers in $[\cdot]$ represent points.

1. Answer the following questions. In YES/NO questions **verify your reasoning**.

   (a) **[1]** $f(t) = \begin{pmatrix} 2\sin t - 3 \\ 2\cos t + 4 \end{pmatrix}$, $t \in [0, 2\pi]$, is a circle. YES/NO

   Solution: YES. This is a circle with the center $(-3, 4)$ and radius 2:

   $$(x+3)^2 + (y-4)^2 = (2\sin t)^2 + (2\cos t)^2 = 4.$$

   (b) **[2]** $f(\varphi_1, \varphi_2, \varphi_3) = (\sin\varphi_2 \cos\varphi_1, \cos\varphi_2, \sin\varphi_2 \sin\varphi_1 \cos\varphi_3, \sin\varphi_2 \sin\varphi_1 \sin\varphi_3)$, $\varphi_1, \varphi_2 \in [0, \pi]$, $\varphi_3 \in [0, 2\pi]$ is a parametrization of a sphere in $\mathbb{R}^4$. YES/NO

   Solution: YES. This is a sphere in $\mathbb{R}^4$ centered at the origin with radius 1:

   $$\begin{aligned} & x^2 + y^2 + z^2 \\ &= (\sin\varphi_2 \cos\varphi_1)^2 + \cos\varphi_2^2 + (\sin\varphi_2 \sin\varphi_1 \cos\varphi_3)^2 + (\sin\varphi_2 \sin\varphi_1 \sin\varphi_3)^2 \\ &= \sin\varphi_2^2 \cos\varphi_1^2 + \cos\varphi_2^2 + \sin\varphi_2^2 \sin\varphi_1^2 \cos\varphi_3^2 + \sin\varphi_2^2 \sin\varphi_1^2 \sin\varphi_3^2 \\ &= \sin\varphi_2^2 \cos\varphi_1^2 + \cos\varphi_2^2 + \sin\varphi_2^2 \sin\varphi_1^2 (\cos\varphi_3^2 + \sin\varphi_3^2) \\ &= \cos\varphi_2^2 + \sin\varphi_2^2 (\cos\varphi_1^2 + \sin\varphi_1^2) \\ &= \cos\varphi_2^2 + \sin\varphi_2^2 = 1. \end{aligned}$$

   (c) **[1]** There exists an analytic solution to the differential equation

   $$y'(x) = (x^3 + 3\sin x + x^2) e^{y^2}.$$

   YES/NO

   Solution: NO. This is an ODE with separable variables:

   $$e^{-y^2} dy = (x^3 + 3\cos x + x^2) dx$$

   Therefore we would need to know the analytic expression for the indefinite integral of e^{-y^2}, to know the analytic solution to the ODE. But this does not exist.

   (d) **[1]** The translation of the second order ODE $x'' - 4x' + x = 0$ into first order system of ODEs is

   $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

   YES/NO

   Solution: YES. We introduce new variables $x_1 = x$ and $x_2 = x'$ to obtain a system $\dot{x}_1 = x_2$, $\dot{x}_2 = 4x_2 - x_1$. The matricial version of this system is

   $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(e) **[1]** Let

$$\dot{x}_1 = x_1 + 2x_2,$$
$$\dot{x}_2 = 2x_1 - 6x_2$$

by a system of differential equations. Then $\lim_{t\to\infty} x_1(t) = 0$ independently of the initial conditions $x_1(0)$, $x_2(0)$. YES/NO

Solution: NO. We only need to compute the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -6 \end{pmatrix}$. The characteristic polynomial is $(1-x)(-6-x) - 4 = x^2 + 5x - 10$ and hence $\lambda_1 = \frac{-5+\sqrt{25+40}}{2} > 0$, $\lambda_2 = \frac{-5-\sqrt{25+40}}{2} < 0$. Hence, the solutions to the system are

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2,$$

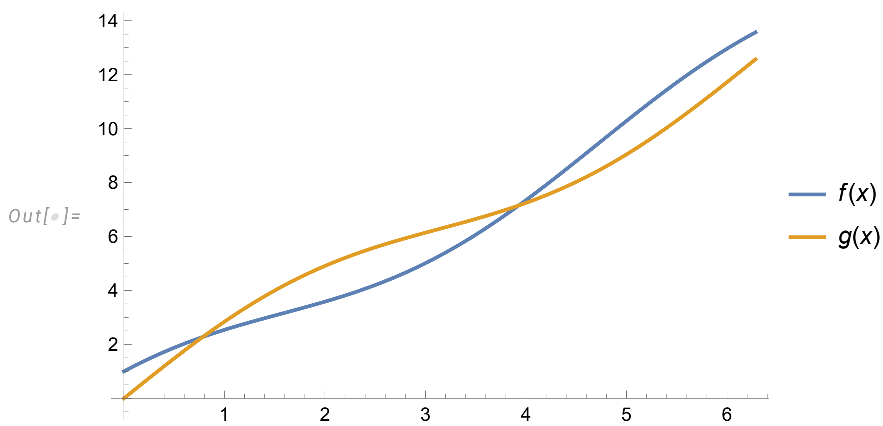
where C_1, C_2 are constants and v_1, v_2 the eigenvectors of A. Since $\lambda_2 < 0 < \lambda_1$, it follows that $\lim_{t\to\infty} x_1(t) = \infty$ for $C_1 > 0$.

2. (a) **[2]** Sketch the graphs of the functions

$$f(x) = 2x + \cos(x) \quad \text{and} \quad g(x) = 2x + \sin(x)$$

for $x \in [0, 2\pi]$. Determine the local extrema of f, g on $[0, 2\pi]$. (You do not need to determine regions of convexity/concavity.)

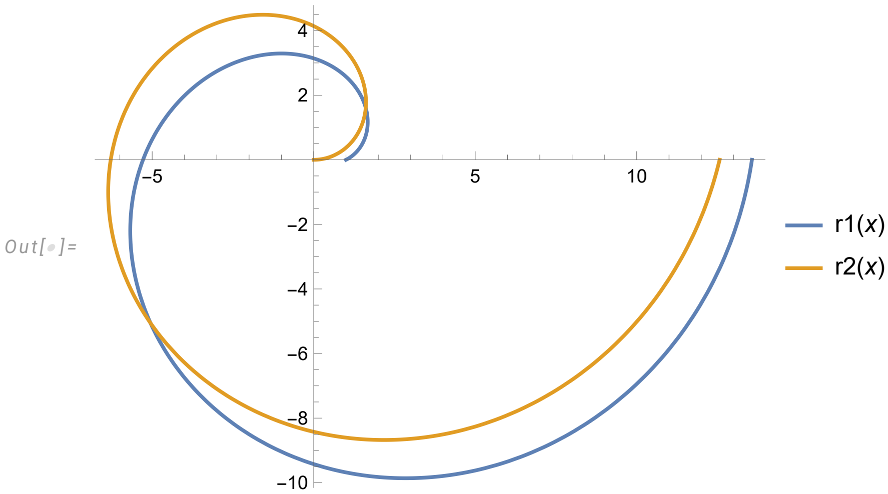
Solution: Since $f'(x) = 2 - \sin(x)$ and $g'(x) = 2 + \cos(x)$, we have that $f'(x) \geq 0$, $g'(x) \geq 0$ for every x. So f and g are both increasing functions on $[0, 2\pi]$. The candidates for extrema are $f'(x) = 0$ and $g'(x) = 0$. But such solutions do not exist and there are no extrema of f, g (except the boundary points).



(b) **[3]** Sketch the closed curves given in polar coordinates by

$$r_1(\varphi) = 2\varphi + \cos\varphi \quad \text{and} \quad r_2(\varphi) = 2\varphi + \sin\varphi.$$

Solution:



(c) **[5]** Compute the area of the bounded region determined by the curves on the interval $\varphi \in [0, 2\pi]$. **Hint:** $\cos^2\varphi = \frac{1+\cos(2\varphi)}{2}$.

Solution: We need to determine the points of intersection of the curves for $\varphi \in [0, 2\pi]$. We have that

$$r_1(\varphi) = r_2(\varphi) \iff \cos(\varphi) = \sin(\varphi) \iff \tan(\varphi) = 1 \iff \varphi \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}.$$

So the area is

$$\begin{aligned}
A &= \frac{1}{2}\left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (r_2^2 - r_1^2)d\varphi\right) \\
&= \frac{1}{2}\left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (4\varphi^2 + 4\varphi\sin\varphi + \sin^2\varphi - 4\varphi^2 - 4\varphi\cos\varphi - \cos^2\varphi)d\varphi\right) \\
&= \frac{1}{2}\left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (4\varphi\sin\varphi + \sin^2\varphi - 4\varphi\cos\varphi - \cos^2\varphi)d\varphi\right) \\
&= \frac{1}{2}\left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (4\varphi\sin\varphi - 4\varphi\cos\varphi + 1 - 2\cos^2\varphi)d\varphi\right) \\
&= \frac{1}{2}\left(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (4\varphi\sin\varphi - 4\varphi\cos\varphi - \cos 2\varphi)d\varphi\right).
\end{aligned}$$

Since

$$\begin{aligned}
\int_a^b \varphi\sin\varphi d\varphi &= [-\varphi\cos\varphi]_a^b + \int_a^b \cos\varphi d\varphi = [-\varphi\cos\varphi + \sin\varphi]_a^b, \\
\int_a^b \varphi\cos\varphi d\varphi &= [\varphi\sin\varphi]_a^b - \int_a^b \sin\varphi d\varphi = [\varphi\sin\varphi + \cos\varphi]_a^b,
\end{aligned}$$

we get

$$\begin{aligned}
A &= 2\left(\frac{5\pi}{4}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - 2\left(-\frac{5\pi}{4}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \\
&\quad - \frac{1}{4}[\sin(2\varphi)]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= 3\pi\sqrt{2} - \frac{1}{4}(1-1) = 3\pi\sqrt{2}.
\end{aligned}$$

3. Let

$$y' = -2xy + e^{-x^2+2x}, \quad y(0) = 1$$

be the DE.

(a) **[4]** Solve the DE explicilty. Solution:
Homogeneous part: $y' = -2xy$. Then $\frac{dy}{y} = -2xdx$ and hence $\log|y| = -x^2 + C$, $C \in \mathbb{R}$. Expressing y we get $y = Ae^{-x^2}$, $A \in \mathbb{R}$.

Particular solution: We use variation of constants: $y_p(x) = A(x)e^{-x^2}$. Hence, $A'(x)e^{-x^2} - 2xA(x)e^{-x^2} = -2xA(x)e^{-x^2} + e^{-x^2+2x}$. Further on, $A'(x)e^{-x^2} = e^{-x^2+2x}$ and so $A'(x) = e^{2x}$. Then $A(x) = \frac{1}{2}e^{2x}$ and $y_p(x) = \frac{1}{2}e^{-x^2+2x}$.

So $y(x) = Ae^{-x^2} + \frac{1}{2}e^{-x^2+2x}$. Using $y(0) = 1$ we get $A = \frac{1}{2}$ and hence $y(x) = \frac{1}{2}e^{-x^2} + \frac{1}{2}e^{-x^2+2x}$.

(b) **[4]** Use Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & & \\ \frac{1}{2} & \frac{1}{2} & 0 & \\ 1 & -1 & 2 & 0 \\ \hline & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array},$$

and the step-size $h = 0.1$ to compute the approximation $y_1 \approx y(0.1)$.
Solution: We have

$$y(0.1) \approx y(0) + \frac{1}{6}k_1 + \frac{4}{6}k_2 + \frac{1}{6}k_3,$$

where

$$\begin{aligned} k_1 &= 0.1 \cdot f(0, y(0)) = 0.1 \cdot f(0, 1) = 0.1 \cdot (0 + e^0) = 0.1, \\ k_2 &= 0.1 \cdot f(\frac{1}{2} \cdot 0.1, y(0) + \frac{1}{2}k_1) = 0.1 \cdot f(0.05, 1 + 0.05) \\ &= 0.1 \cdot (-2 \cdot 0.05 \cdot 1.05 + e^{-0.05^2+0.1}) \approx 0.0997, \\ k_3 &= 0.1\dot{f}(0.1, y(0) - k_1 + 2k_2) = 0.1 \cdot f(0.1, 1 - 0.1 + 0.0997) \\ &= 0.1 \cdot (-2 \cdot 0.1 \cdot 0.9997 + e^{-0.1^2+2 \cdot 0.1}) \approx 0.0989. \end{aligned}$$

Finally,

$$y(0.1) \approx 1 + \frac{1}{6} \cdot 0.1 + \frac{4}{6} \cdot 0.0997 + \frac{1}{6} \cdot 0.0989 = 1.0996.$$

(c) **[1]** Estimate the error of the numerical solution of $y(0.1)$.
Solution: Error is $|\frac{1}{2}e^{-0.1^2} + \frac{1}{2}e^{-0.1^2+2 \cdot 0.1} - 1.0996| \approx 0.005986$.