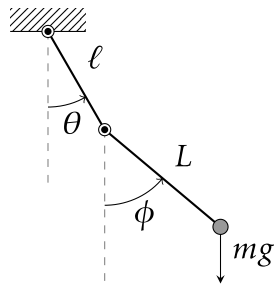
Forced oscillations

A *mathematical pendulum* is a pendulum at which a point mass m hangs on light, straight (and rigid) rod of length L which can freely turn around the pivot point. The mass m is under influence of the gravitational force mg, denote the displacement angle at time t by $\phi(t)$. Clamping this pendulum to the end of another light rod of length ℓ connected to a drive shaft under the ceiling gives us the mechanical system on the right.



A differential equation for the angle of displacement ϕ

For the usual mathematical pendulum (ie. $\theta = \text{const.}$) the function ϕ is a solution of the differential equation

$$L\ddot{\phi} + g\sin(\phi) = 0. \tag{1}$$

This differential equation is only valid for the case of a pendulum which can rotate without friction around the pivot point and there are no external influences. We'll add an external influence: Clamp the pendulum freely to the end of a rotating rod of length ℓ, which is in turn clamped rigidly to a drive shaft. The drive shaft rotates this rod around its axis for the angle $\theta = \theta(t)$. The differential equation for the displacement angle ϕ from the equilibrium position now becomes

$$L\ddot{\phi} + g\sin(\phi) = -\ell\ddot{\theta}\cos(\theta - \phi) + \ell\dot{\theta}^2\sin(\theta - \phi). \tag{2}$$

Note that, in this equation, $\theta(t)$ is a given function, while $\phi(t)$ is the unknown function.

Add a moving target to this system – a target which evenly approaches our pendulum in the direction perpendicular to the plane of oscillation. We know the moment at which the target will reach the plane of oscillation, and we would like to achieve that the point mass at the bottom of the pendulum hits a prescribed vertical line (given by $x = x_0$) on the target. By how much do we have to displace the pendulum out of its equilibrium position at the moment the target begins to approach the pendulum so that the pendulum will hit the prescribed vertical line on the target?

Two Octave functions will give us the answer. The first one will use the Runge–Kutta method of order 4 to solve a system of differential equations of the 1$^{\text{st}}$ order, which we obtain from equation (2). We'll input initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = \omega_0$ at the moment $t = 0$, and get the position

$[x,y]^\mathsf{T}$ of the point mass at the moment $t = T$ as the moving target reaches the plane of oscillation. We have just described a vector–valued function

$$\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{F}([\phi_0, \omega_0]^\mathsf{T}) = [x,y]^\mathsf{T}.$$

The composite $\mathbf{F}([\phi_0, 0]^\mathsf{T}) = [x,y]^\mathsf{T} \mapsto x$ now describes a function $f : \mathbb{R} \to \mathbb{R}$, $f(\phi_0) = x$. The second of our Octave functions will use a freely chosen variant of the Newton's method to solve the equation $f(\phi_0) = x_0$ in the unknown ϕ_0. We will assume the following for our pendulum:

$$g = m = \ell = L = 1,\ -1 \le x_0 \le 1.$$

Task

1. Use Newton's second law $\mathbf{F} = m\mathbf{a}$ to derive the equation (1).

2. Write down the formulae for the coordinates x any y of the position of the point mass m depending on angles θ and ϕ, and rod lengths ℓ and L.

3. Rewrite the differential equation (2) as a system of two first order differential equations in functions ϕ and $\omega = \dot{\phi}$.

4. Write an Octave function `pos = pendulum(Phi0, Theta, T)` which, given initial conditions $\boldsymbol{\phi}_0 = [\phi_0, \omega_0]^\mathsf{T}$ and a function

   $$\boldsymbol{\theta}(t) = [\theta(t), \dot{\theta}(t), \ddot{\theta}(t)]^\mathsf{T},$$

   solves the differential equation (2) until the terminal time T and returns the position $[x,y]^\mathsf{T}$ of the point mass at the time T.
   Stick to specifications: `Phi0 = [phi0; omega0]` *is a column,* `Theta` *is a function of t, which returns a column* $[\theta(t), \dot{\theta}(t), \ddot{\theta}(t)]^\mathsf{T}$*,* `T` *is a number,* `pos = [x; y]` *is a column.*

5. Write an Octave function `phi0 = onTarget(x0, Theta, T)` which, given the x-component of the terminal position x_0, returns the initial displacement angle ϕ_0, at which (with initial speed $\omega_0 = 0$) after elapsed time T the mass point on the pendulum hits the vertical given by $x = x_0$ on the target.
   Stick to specifications: `x0`, `T` *and* `phi0` *are numbers,* `Theta` *as above.*

Submission

Use the online classroom to submit the following:

1. files **pendulum.m** and **onTarget.m**, which should be well-commented and contain at least one test,
2. a report file **solution.pdf** which contains the necessary derivations and answers to questions.

While you can discuss solutions of the problems with your colleagues, the programs and report must be your own creation. You can use all Octave functions from problem sessions (eg. `rk4.m` and one of `dnewton.m`, `secant.m` or `broyden.m`).