1. Let

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -2 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}, \quad A' = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 1 & 2 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 0 \end{bmatrix}.$$

   (a) Is the system $A\mathbf{x} = \mathbf{b}$ solvable? Is the system $A'\mathbf{x} = \mathbf{b}$ solvable? Find orthogonal projections $\mathbf{b}_1$ and $\mathbf{b}_1'$ of the vector $\mathbf{b}$ onto $C(A)$ and $C(A')$, and then find all the solutions of the systems $A\mathbf{x} = \mathbf{b}_1$ and $A'\mathbf{x} = \mathbf{b}_1'$.

   (b) Find the singular value decomposition of A; $A = USV^\mathsf{T}$. This can be obtained using the eigenvalue decomposition of $A^\mathsf{T}A$.

   (c) Find the Moore–Penrose pseudoinverses of A and A', and evaluate $A^+\mathbf{b}$ and $A'^+\mathbf{b}$. Explain the result.

   (d) Solve the exercise in octave, using the commands `svd(A)` and `pinv(A)`.

2. **SVD and image compression.** A greyscale image can be represented by a matrix A. (A color image can be represented using three matrices, say A_R, A_G and A_B). Using the matrices U, S, and V from the SVD decomposition we can reconstruct the matrix A by computing USV^T. Moreover, we can decide that small singular values contribute very little to the image and can be ignored. Let S' be the matrix that contains the largest m singular values on the diagonal. Then $A' = US'V^\mathsf{T}$ can serve as an approximation to A.

   (a) Download the image `lena512.mat` and use `A = imread("lena512.mat")` to load it into octave/Matlab. To show the image use `imshow(A)`.

   (b) Find the SVD decomposition of A.

   (c) Compute the approximations for A obtained by using 10, 20, 50, 100 of the largest singular values of A. Show the images and visually asses the quality of the images.

   (d) How much space would we actually need to save such an approximation?