Linearna algebra

1. Izračunaj spodnje determinante

(a) $\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 1\cdot 7 - 2\cdot 3 = 7 - 6 = \underline{\underline{1}}$

(b) $\begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = 2\cdot 1 - 0\cdot 2 = \underline{\underline{2}}$

(c) $\begin{vmatrix} 0 & -1 & 0 & 3 \\ -2 & 0 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{vmatrix} = -(-1)\begin{vmatrix} -2 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \end{vmatrix} - 3\begin{vmatrix} -2 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & -1 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} - 3\cdot(-2)\begin{vmatrix} 0 & -1 \\ 2 & -1 \end{vmatrix} =$

$= -2\cdot(-1) + 6\cdot 2 = 2 + 12 = \underline{\underline{14}}$

(d) $\begin{vmatrix} -4 & 3 & 2 & -2 \\ 5 & -1 & -2 & -3 \\ 2 & 0 & -4 & -5 \\ -5 & 3 & -2 & 1 \end{vmatrix} \overset{\substack{\mathrm{I}+2\mathrm{II} \\ \mathrm{IV}+\mathrm{II}}}{=} \begin{vmatrix} 0 & 3 & -6 & -12 \\ 5 & -1 & -2 & -3 \\ 2 & 0 & -4 & -5 \\ 0 & 2 & -4 & -2 \end{vmatrix} = 3\cdot 2 \begin{vmatrix} 0 & 1 & -2 & -4 \\ 5 & -1 & -2 & -3 \\ 2 & 0 & -4 & -5 \\ 0 & 1 & -2 & -1 \end{vmatrix} \overset{\mathrm{IV}-\mathrm{I}}{=}$

$= 6\begin{vmatrix} 0 & 1 & -2 & -4 \\ 5 & -1 & -2 & -3 \\ 2 & 0 & -4 & -5 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 6\cdot 3\begin{vmatrix} 0 & 1 & -2 \\ 5 & -1 & -2 \\ 2 & 0 & -4 \end{vmatrix} = 18\cdot 2\begin{vmatrix} 0 & 1 & -2 \\ 5 & -1 & -2 \\ 1 & 0 & -2 \end{vmatrix} \overset{\substack{\mathrm{II}-\mathrm{I} \\ \mathrm{III}-\mathrm{I}}}{=} 36\begin{vmatrix} 0 & 1 & -2 \\ 5 & -2 & 0 \\ 1 & -1 & 0 \end{vmatrix}$

$= 36\cdot(-2)\begin{vmatrix} 5 & -2 \\ 1 & -1 \end{vmatrix} = -72(-5+2) = 72\cdot 3 = \underline{\underline{216}}$

- vrstici lahko prištejemo večkratnik druge vrstice
- če zamenjamo 2 vrstici ali 2 stolpca, se rezultat pomnoži z -1
- lahko izpostavimo skalar iz vrstice ali stolpca

2. Za katere vrednosti parametrov x oziroma a spodnji matriki *nimata* inverza?

(a) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ x^3 & x & 0 & x^2 \\ 0 & 0 & x-1 & 0 \\ -1 & 0 & -3 & x+1 \end{bmatrix}$

$\boxed{\begin{array}{l} A \text{ je obrnljiva} \Leftrightarrow \det A \neq 0 \\ A^{-1} \text{ ne obstaja} \Leftrightarrow \det A = 0 \end{array}}$

$\begin{vmatrix} 1 & 0 & 3 & 0 \\ x^3 & x & 0 & x^2 \\ 0 & 0 & x-1 & 0 \\ -1 & 0 & -3 & x+1 \end{vmatrix} = (x-1)\begin{vmatrix} 1 & 0 & 0 \\ x^3 & x & x^2 \\ -1 & 0 & x+1 \end{vmatrix} = (x-1)x\begin{vmatrix} 1 & 0 \\ -1 & x+1 \end{vmatrix} = (x-1)x(x+1) = 0$

za $x \in \{0, 1, -1\}$ ni inverza

(b) $$\begin{vmatrix} -1 & a & a^2-1 & 1 \\ 0 & a & -1 & 1 \\ -1 & -a & 1-a^2 & 0 \\ 1 & 0 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & a & a^2-2 & 0 \\ 1 & a & -2 & 0 \\ -1 & -a & 1-a^2 & 0 \\ 1 & 0 & -1 & -1 \end{vmatrix} = -\begin{vmatrix} 0 & a & a^2-2 \\ 1 & a & -2 \\ -1 & -a & 1-a^2 \end{vmatrix} =$$

$$= -\begin{vmatrix} 0 & a & a^2-2 \\ 1 & a & -2 \\ 0 & 0 & -1-a^2 \end{vmatrix} = +\begin{vmatrix} a & a^2-2 \\ 0 & -1-a^2 \end{vmatrix} = -\begin{vmatrix} a & a^2-2 \\ 0 & 1+a^2 \end{vmatrix} = -a(a^2+1) = 0$$

za $a=0$ ni inverza

3. Izračunaj spodnje determinante ali pa vsaj poišči rekurzivno zvezo, ki jih izraža.

(a) $$\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 1 \end{vmatrix} n = d_n$$

n

d_{n-1}

$$d_1 = \begin{vmatrix} 1 \end{vmatrix} = 1$$

$$d_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$d_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} =$$

$$= d_2 - \begin{vmatrix} 1 \end{vmatrix} = d_2 - d_1 = 0 - 1 = -1$$

$$d_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = d_3 - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = d_3 - d_2 = -1 - 0 = -1$$

$\vdots$

$$\underline{\underline{d_n = d_{n-1} - d_{n-2}}}$$

n	1	2	3	4	5	6	7	8	9	10	...
d_n	1	0	-1	-1	0	1	1	0	-1	-1	...

ta vzorec se ponavlja

(b) $$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 & 1 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix} = d_n$$

d_{n-1}

$$d_1 = \begin{vmatrix} 1 \end{vmatrix} = 1$$

$$d_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$d_3 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = d_2 - 1 = 0 - 1 = -1$$

$$= d_2 - d_1 = 0 - 1 = -1$$

$$d_4 = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = d_3 - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = d_3 - \det I_2 = -1 - 1 = -2$$

$$d_5 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = d_4 - \det I_3 = -2 - 1 = -3$$

$\vdots$

$d_n = d_{n-1} - \det I_{n-2}$

$\underline{\underline{d_n = d_{n-1} - 1}}$

n	1	2	3	4	5	6	...
d_n	1	0	−1	−2	−3	−4	...

$\underline{\underline{d_n = 2 - n}}$ (dokažemo z indukcijo)

4. Iz matrik $A, B \in \mathbb{R}^{n \times n}$ sestavimo $2n \times 2n$ bločno matriko

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

Prepričaj se, da velja formula

$$\det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \det(A + B) \cdot \det(A - B).$$

Ali je ta determinanta enaka $\det(A^2 - B^2)$? Utemelji ali pa poišči protiprimer!

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A & B \\ B + A & A + B \end{vmatrix} = \begin{vmatrix} A - B & B \\ 0 & A + B \end{vmatrix} = \det(A - B) \cdot \det(A + B)$$

$2n$ $2n$

$(n+i)$-ti vrstici prištejemo i-to vrstico (za $i = 1, \dots, n$)

i-tem stolpcu odštejemo $(n+i)$-ti stolpec (za $i = 1, \dots, n$)

determinanta (bločno) zgornjetrikotne matrike je produkt (determinant) diagonalnih (bločnih) elementov

Ampak $\det(A-B) \cdot \det(A+B) \neq \det(A^2 - B^2)$.

Npr.: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A - B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $A^2 - B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$L = (0 + 1)(0 - 1) = -1$, $D = 0 \Rightarrow L \neq D$

5. Dani sta matriki

$$S=\begin{bmatrix}2&1&0&0\\1&2&1&0\\1&1&2&1\\1&1&1&2\end{bmatrix}\quad\text{in}\quad T=\begin{bmatrix}2&1&1&1\\1&2&1&1\\0&1&2&1\\0&0&1&2\end{bmatrix}.$$

Izračunaj determinante matrik S, T, ST, ST^{-1} ter $(S-T)^{-1}$.

$$\det S = \begin{vmatrix}2&1&0&0\\1&2&1&0\\1&1&2&1\\1&1&1&2\end{vmatrix} = \begin{vmatrix}2&1&0&0\\1&2&1&0\\1&1&2&1\\-1&-1&-3&0\end{vmatrix} = +\begin{vmatrix}2&1&0\\1&2&1\\1&1&3\end{vmatrix} = \begin{vmatrix}0&1&0\\-3&2&1\\-1&1&3\end{vmatrix} = -\begin{vmatrix}-3&1\\-1&3\end{vmatrix} = \begin{vmatrix}3&1\\1&3\end{vmatrix} =$$

IV − 2III −1 I − 2II

$$= 3\cdot 3 - 1 = \underline{\underline{8}}$$

$$T = S^T \Rightarrow \det T = \det S = 8$$

$$\det(ST) = \det(S)\cdot\det(T) = 8\cdot 8 = 64$$

$$\det(ST^{-1}) = \det(S)\cdot\det(T^{-1}) = \det(S)\cdot(\det(T))^{-1} = \frac{\det S}{\det T} = \frac{8}{8} = 1$$

$$\det\left((S-T)^{-1}\right) = \left(\det(S-T)\right)^{-1} = ? \qquad = \begin{vmatrix}0&0&-1&-1\\0&0&0&-1\\1&0&0&0\\1&1&0&0\end{vmatrix} = \begin{vmatrix}0&0&1&1\\0&0&0&1\\1&0&0&0\\1&1&0&0\end{vmatrix} = \begin{vmatrix}0&1&1\\0&0&1\\1&0&0\end{vmatrix} = 1\cdot 1 - 0\cdot 1 = 1$$

6. Naj bosta $\mathbf{x}$ in $\mathbf{y}$ poljubna vektorja iz $\mathbb{R}^n$.

(a) Izrazi determinanto matrike

$$\begin{bmatrix}1&-\mathbf{y}^{\mathsf T}\\\mathbf{x}&I\end{bmatrix}$$

z enostavno formulo $\mathbf{x}$ in $\mathbf{y}$.

$$\begin{array}{c}\\ y_1\cdot\rightarrow\\ y_2\cdot\rightarrow\\ \\ \\ y_n\cdot\rightarrow\end{array}\begin{vmatrix}1&-y_1&-y_2&-y_3&\cdots&-y_n\\x_1&1&&&&\\x_2&&1&&&\\&&&1&&\\&&&&1&\\x_n&&&&&1\end{vmatrix} = \begin{vmatrix}1-x_1y_1-\cdots-x_ny_n&0&0&0&\cdots&0\\x_1&1&&&&\\x_2&&1&&&\\x_3&&&1&&\\\vdots&&&&1&\\x_n&&&&&1\end{vmatrix} =$$

$\mathrm{I} + y_1\mathrm{II} + y_2\mathrm{III} + \cdots$

$$= (1-\vec{x}\cdot\vec{y})\cdot\det I_n = 1-\vec{x}\cdot\vec{y} = \underline{\underline{1-x^T\vec{y}}}$$

(b) Kako bi izračunal $\det(I + \mathbf{x}\mathbf{y}^\top)$? Koliko je $\det(I + \mathbf{x}\mathbf{y}^\top)$, če sta $\mathbf{x}$ in $\mathbf{y}$ pravokotna vektorja?

$$\det(I + xy^T) = \det\left(I + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}\right) = \det\left(I + \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & & & x_2y_n \\ x_3y_1 & & & & \\ \vdots & & & & \\ x_ny_1 & x_ny_2 & & \cdots & x_ny_n \end{bmatrix}\right) =$$

$$= \begin{vmatrix} 1+x_1y_1 & x_1y_2 & x_1y_3 & \cdots & x_1y_n \\ x_2y_1 & 1+x_2y_2 & x_2y_3 & \cdots & x_2y_n \\ x_3y_1 & x_3y_2 & 1+x_3y_3 & \cdots & x_3y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & x_ny_3 & \cdots & 1+x_ny_n \end{vmatrix}$$

Iz a): $1 - x^Ty = \begin{vmatrix} 1 & -\vec{y}^T \\ \vec{x} & I \end{vmatrix} = \begin{vmatrix} 1 & -y_1 & -y_2 & -y_3 & \cdots & -y_n \\ x_1 & 1 & & & & \\ x_2 & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ x_n & & & & & 1 \end{vmatrix} \overset{+}{=} \begin{vmatrix} 1 & -y_1 & -y_2 & \cdots & -y_n \\ 0 & 1+x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ 0 & x_2y_1 & 1+x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_ny_1 & x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix} =$

$\mathrm{II} - x_1\mathrm{I}$
$\mathrm{III} - x_2\mathrm{I}$
$\vdots$

$$= 1 \cdot \det(I + xy^T)$$

$$\Rightarrow \underline{\underline{\det(I + xy^T) = 1 - x^Ty = 1 - y^Tx}}$$

$$\vec{x} \cdot \vec{y} = \vec{x}^T\vec{y} = y^T\vec{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in \mathbb{R}$$

skalarni produkt $\approx$ matrično množenje