1.1.1. OPERACIJE Z VEKTORJI

(C) SKALARNI PRODUKT $\quad \vec{x}\cdot\vec{y} \in \mathbb{R}$

Def: Za $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ je

$$(\vec{x}\,\vec{y}) = \vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

skalarni produkt vektorjev $\vec{x}$ in $\vec{y}$.

Lastnosti skalarnega produkta:

(1) $\boxed{\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}}$ (komutativnost / simetričnost)

(2) $$(\alpha\vec{x})\cdot\vec{y} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \underset{\text{def}}{\overset{\uparrow}{=}} \alpha x_1y_1 + \alpha x_2y_2 + \dots + \alpha x_ny_n = \alpha(x_1y_1 + \dots + x_ny_n) = \alpha(\vec{x}\cdot\vec{y})$$

$$\boxed{(\alpha\vec{x})\cdot\vec{y} = \alpha(\vec{x}\cdot\vec{y}) = \vec{x}\cdot(\alpha\vec{y})}$$

(homogenost) $\vec{x},\vec{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$

(3) $\vec{x},\vec{y},\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^n$

$$\vec{x}\cdot(\vec{y}+\vec{z}) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1+z_1 \\ y_2+z_2 \\ \vdots \\ y_n+z_n \end{bmatrix} \overset{\text{def}}{=} x_1(y_1+z_1) + \dots + x_n(y_n+z_n) \overset{\text{distr. v } \mathbb{R}}{=}$$

$$= \underline{x_1y_1} + \underline{x_1z_1} + \dots + \underline{x_ny_n} + \underline{x_nz_n} \overset{\text{def}}{=}$$

$$= \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$$

$$\boxed{\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}}$$ aditivnost

linearnost

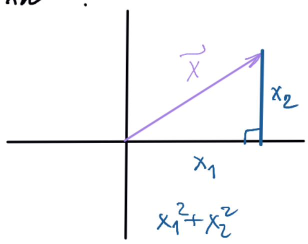
(4) $\vec{x}\cdot\vec{x} = x_1\cdot x_1 + \dots + x_nx_n = \underset{\overset{\vee|}{0}}{x_1^2} + \dots + \underset{\overset{\vee|}{0}}{x_n^2}$

$$\boxed{\vec{x}\cdot\vec{x} \geq 0}$$

$$\vec{x}\cdot\vec{x} = 0 \iff x_1^2 + \dots + x_n^2 = 0 \iff x_1 = \dots = x_n = 0 \iff \vec{x} = \vec{0}$$

$$\boxed{\vec{x}\cdot\vec{x} = 0 \iff \vec{x} = \vec{0}}$$ (pozitivna definitnost)

Def: Dolžina vektorja $\vec{x}\in\mathbb{R}^n$ je $\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2+\dots+x_n^2}$.
(ali norma)



Primer: $\vec{x}=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$

$$\vec{x}\cdot\vec{y} = 1\cdot 2 + 2\cdot 1 + 3\cdot(-1) = 1$$

$$\|\vec{x}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$
$$\|\vec{y}\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$$

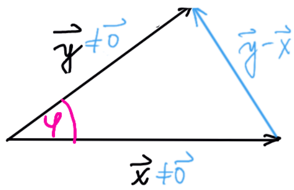
$\frac{1}{\sqrt{14}}\vec{x}$ je enotski vektor.

Def: Vektor $\vec{x}\in\mathbb{R}^n$ je enotski (normiran), če je $\|\vec{x}\|=1$.

Vsak vektor $\frac{\vec{x}}{\|\vec{x}\|}$ je enotski.

Edini vektor dolžine 0 je $\vec{0}$.

Kot med vektorjema



$$\|\vec{y}-\vec{x}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\varphi \quad \text{(kosinusni izrek)}$$
$$(\vec{y}-\vec{x})(\vec{y}-\vec{x}) = \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} - 2\|\vec{x}\|\|\vec{y}\|\cos\varphi \quad \text{(def dolžine)}$$
$$\cancel{\vec{y}\cdot\vec{y}} - \vec{x}\cdot\vec{y} - \vec{y}\cdot\vec{x} + \cancel{\vec{x}\cdot\vec{x}} = \cancel{\vec{x}\cdot\vec{x}} + \cancel{\vec{y}\cdot\vec{y}} - 2\|\vec{x}\|\|\vec{y}\|\cos\varphi \quad \text{(linearnost)} \quad \Big/\cdot(-1)$$
$$\vec{x}\cdot\vec{y} + \underset{\substack{\| \\ \vec{x}\cdot\vec{y}\ \text{(simetričnost)}}}{\vec{y}\cdot\vec{x}} = 2\|\vec{x}\|\|\vec{y}\|\cos\varphi$$

$$2\,\vec{x}\cdot\vec{y} = 2\|\vec{x}\|\|\vec{y}\|\cos\varphi \quad \Big/ :2$$

$$\boxed{\vec{x}\cdot\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\varphi} \quad \Big/ :(\|\vec{x}\|\|\vec{y}\|)$$

Če $\vec{x},\vec{y}\neq\vec{0}$, potem

$$\boxed{\cos\varphi = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}} \quad \leftarrow \text{tako računamo kote med vektorji}$$

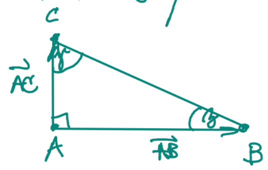
Lastnosti:
1) Če $\varphi\in[0,\frac{\pi}{2})$, potem $\cos\varphi>0$, in zato $\vec{x}\cdot\vec{y}>0$
2) $\varphi\in(\frac{\pi}{2},\pi]$, potem $\cos\varphi<0$, in zato $\vec{x}\cdot\vec{y}<0$.
3) Če $\varphi=\frac{\pi}{2}$, je $\cos\frac{\pi}{2}=0$ in zato $\vec{x}\cdot\vec{y}=0$.

Vektorja $\vec{x}$ in $\vec{y}$ sta pravokotna (ortogonalna), če $\vec{x}\cdot\vec{y}=0$.
(Vektor $\vec{0}$ je pravokoten na vsak vektor.)

Primer: Dane so točke $A(1,2,3)$, $B(2,2,1)$ in $C(3,1,c)$ v $\mathbb{R}^3$

a) Določimo c tako, da bo $\triangle ABC$ pravokotni trikotnik s pravim kotom pri oglišču A.

b) Določimo še kot β pri oglišču B.



a) $\vec{AB} \perp \vec{AC} \iff \vec{AB}\cdot\vec{AC} = 0$

$$\iff \begin{bmatrix}1\\0\\-2\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\c-3\end{bmatrix} = 0 \iff 1\cdot 2 + 0\cdot(-1) + (-2)(c-3) = 0$$

$$\iff 2 - 2c + 6 = 0$$

$$\iff 2c = 8$$

$$\iff \underline{c = 4}$$

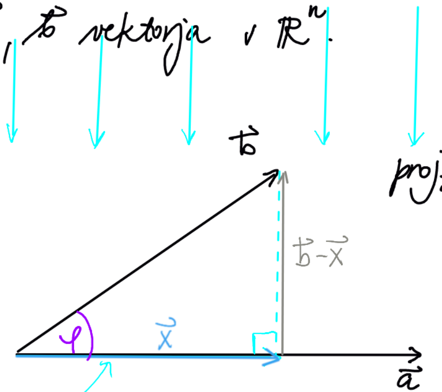
b) $$\cos\beta = \frac{\vec{BA}\cdot\vec{BC}}{\|\vec{BA}\|\cdot\|\vec{BC}\|} = \frac{-1+0+6}{\sqrt{5}\,\sqrt{11}} = \frac{5}{\sqrt{5}\sqrt{11}} \Rightarrow \beta = 47{,}61^\circ$$

$$\vec{BC} = \begin{bmatrix}1\\-1\\3\end{bmatrix},\quad \vec{BA} = \begin{bmatrix}-1\\0\\2\end{bmatrix}$$

Pravokotna projekcija (vektorja na vektor)

- računalniški zaslon
- zemljevidi
- sence
- visokodimenzionalni podatki
- ⋮

Naj $\vec{a} \neq \vec{0}$, $\vec{b}$ vektorja v $\mathbb{R}^n$.



$\mathrm{proj}_{\vec{a}}\vec{b} = \vec{x}$ ... projekcija vektorja $\vec{b}$ na $\vec{a}$ (vzdolž $\vec{b}-\vec{x}$)

Lastnosti:
- $\vec{x} = \alpha\vec{a}$ (L1)
- $\vec{a} \perp (\vec{b}-\vec{x})$ (L2)

(pravokotna) projekcija vektorja $\vec{b}$ na vektor $\vec{a}$ $= \mathrm{proj}_{\vec{a}}\vec{b}$

(L2) $\vec{a}\cdot(\vec{b}-\vec{x}) = 0$

(L1) $\vec{a}\cdot(\vec{b}-\alpha\vec{a}) = 0$

(linearnost) $\vec{a}\cdot\vec{b} - \alpha\,\vec{a}\cdot\vec{a} = 0$

$\alpha\,\vec{a}\cdot\vec{a} = \vec{a}\cdot\vec{b}$ $\quad /: \vec{a}\cdot\vec{a} \neq 0$ (saj $\vec{a}\neq\vec{0}$)

$$\alpha = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} \Rightarrow \boxed{\mathrm{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a}}$$

$$\Rightarrow \|\mathrm{proj}_{\vec{a}}\vec{b}\| = \|\vec{b}\|\,|\cos\varphi| = \frac{|\vec{a}\cdot\vec{b}|}{\|\vec{a}\|}$$

(D) VEKTORSKI PRODUKT

→ le za vektorje v $\mathbb{R}^3$!

$\vec{a} \times \vec{b} \in \mathbb{R}^3$

Def: Za $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ in $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ je

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \in \mathbb{R}^3$$

vektorski produkt vektorjev $\vec{a}$ in $\vec{b}$.

Primer:
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2\cdot 6 - 3\cdot 5 \\ 3\cdot 4 - 1\cdot 6 \\ 1\cdot 5 - 2\cdot 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

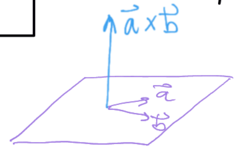
Lastnosti: (1) $\vec{a} \times \vec{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

(2) $\vec{b} \times \vec{a} = \begin{bmatrix} b_2 a_3 - b_3 a_2 \\ b_3 a_1 - b_1 a_3 \\ b_1 a_2 - b_2 a_1 \end{bmatrix} = -(\vec{a} \times \vec{b})$

(3) (geometrijska lastnost)

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} + \cancel{a_2 a_3 b_1} - \cancel{a_1 a_2 b_3} + \cancel{a_1 a_3 b_2} - \cancel{a_2 a_3 b_1} = 0$$

$\boxed{(\vec{a} \times \vec{b}) \perp \vec{a}}$ in (podobno) $\boxed{(\vec{a} \times \vec{b}) \perp \vec{b}}$

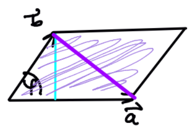


(4) (geometrijska lastnost, 2. del)

$$\boxed{\|\vec{a} \times \vec{b}\| = \underbrace{\|\vec{a}\| \, \|\vec{b}\| \, |\sin\varphi|}_{\text{ploščina paralelograma, napetega na } \vec{a} \text{ in } \vec{b}}}$$

dolžina $(\vec{a} \times \vec{b})$

ploščina paralelograma, napetega na $\vec{a}$ in $\vec{b}$



$pl\, \Delta(\vec{a}, \vec{b}) = \frac{1}{2} \|\vec{a} \times \vec{b}\|$

dokaz: $\|\vec{a} \times \vec{b}\|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 =$

(leva stran)^2 $= a_2^2 b_3^2 + a_3^2 b_2^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_1^2 + a_1^2 b_3^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_1 a_2 b_1 b_2$

$(\|\vec{a}\| \, \|\vec{b}\| \cos\varphi)^2 = (\vec{a} \cdot \vec{b})^2$

(desna stran)^2

$\|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\varphi = \|\vec{a}\|^2 \|\vec{b}\|^2 (1 - \cos^2\varphi) = \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\varphi =$

$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 =$

$= \cancel{a_1^2 b_1^2} + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + \cancel{a_2^2 b_2^2} + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + \cancel{a_3^2 b_3^2} -$
$- (\cancel{a_1^2 b_1^2} + \cancel{a_2^2 b_2^2} + \cancel{a_3^2 b_3^2} + 2a_1 a_2 b_1 b_2 + 2a_1 a_3 b_1 b_3 + 2a_2 a_3 b_2 b_3)$

Iz $\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\varphi$ sledi $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \, |\sin\varphi|$.

V posebnem:

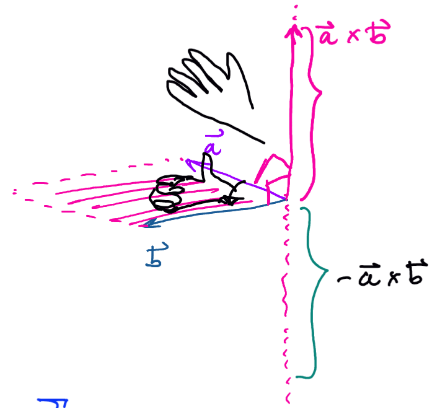
$$\vec a \times \vec b = \vec 0 \iff \|\vec a \times \vec b\| = 0$$

- $\vec a = 0$
- $\vec b = 0$
- $\varphi = k\pi,\ k \in \mathbb{Z} \iff \vec a$ in $\vec b$ kolinearna

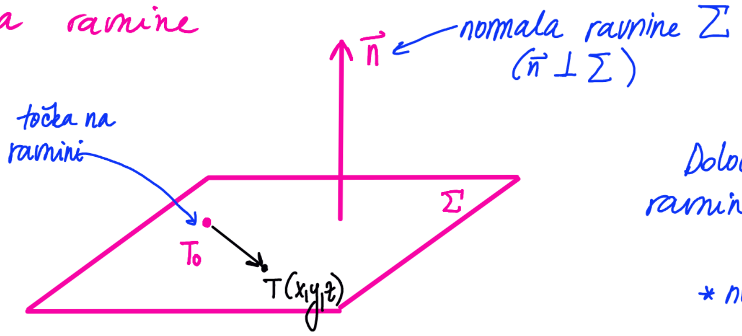
$\vec a \times \vec b = \vec 0 \iff \vec a \parallel \vec b$

$\vec a \cdot \vec b = 0 \iff \vec a \perp \vec b$

(5) (geometrijska lastnost, 3. del) → pravilo desne roke



Enačba ravnine



Določimo enačbo vseh točk na ravnini Σ, če poznamo

* normalo $\vec n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec n \perp \Sigma$
* točko $T_0(x_0, y_0, z_0) \in \Sigma$

Točka $T(x,y,z)$ leži na ravnini Σ, če $\overrightarrow{T_0T} \parallel \Sigma$, tj.

$$\overrightarrow{T_0T} \perp \vec n \Rightarrow \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

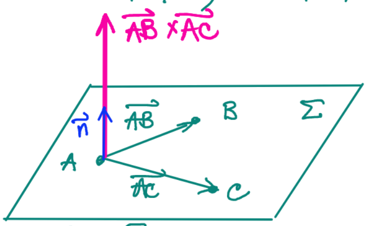
$$\Rightarrow ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{= d}$$

koordinate točke na ravnini

koordinate normale

Primer: Določimo enačbo ravnine, ki vsebuje točke $A(-1,2,-1)$, $B(2,-1,2)$ in $C(0,0,-1)$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0-(-6) \\ 3-0 \\ -6-(-3) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$



$$\vec n = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \left(\text{ali pa } \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} \text{ ali pa } \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \text{ ali pa } \begin{bmatrix} 2\pi \\ \pi \\ -\pi \end{bmatrix}, \ldots\right)$$

$$\underline{2x + y - z = 1}$$

če vstavimo C

$$d = 2 \cdot 0 + 0 - (-1) = 1$$

$$= 2 \cdot (-1) + (2) - (-1) = 1$$

če vstavimo A

(C+D) Mešani produkt

$\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3, \quad \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\boxed{(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})}$$

dokaz: $(\vec{a} \times \vec{b}) \cdot \vec{c} = (a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1)$

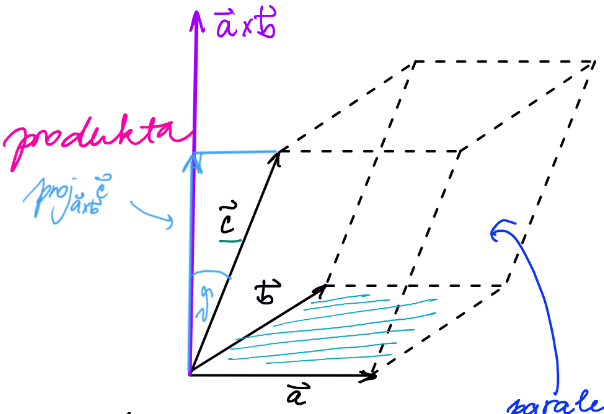
Def: Mešani produkt vektorjev $\vec{a}, \vec{b}, \vec{c}$ je enak

$$(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) \in \mathbb{R}$$

Velja $(\vec{a}, \vec{b}, \vec{c}) = (\vec{b}, \vec{c}, \vec{a}) = (\vec{c}, \vec{a}, \vec{b}) =$
$= -(\vec{a}, \vec{c}, \vec{b}) = -(\vec{b}, \vec{a}, \vec{c}) = -(\vec{c}, \vec{b}, \vec{a})$.

Geometrijski pomen mešanega produkta

$\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$



paralelepiped (= 3D paralelogram)

$$|(\vec{a}, \vec{b}, \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}| = \|\vec{a} \times \vec{b}\| \cdot \underbrace{\|\vec{c}\| \cdot |\cos \vartheta|}_{\substack{\| \\ \|\text{proj}_{\vec{a} \times \vec{b}} \vec{c}\| \\ \| \\ \text{višina paralelepipeda}}} \qquad \vartheta = \sphericalangle(\vec{a} \times \vec{b}, \vec{c})$$

– prostornina paralelepipeda, napetega na $\vec{a}, \vec{b}$ in $\vec{c}$.

Posledica: $(\vec{a}, \vec{b}, \vec{c}) = 0 \iff \vec{a}, \vec{b}, \vec{c}$ ležijo v isti ravnini.