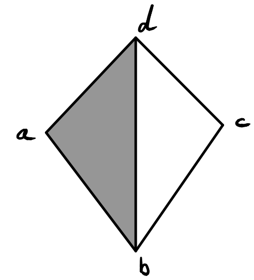
Homology

1. Chains

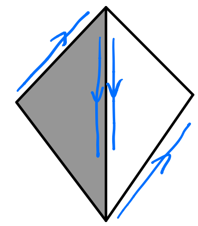
- K a scx of dim n.



- Choose a Abelian group G for coefficients ($\mathbb{Z}, \mathbb{Z}_p, \mathbb{R}$)
- $n_p = \#$ of p-sxes of K, $\forall p \in \{0, 1, \dots, n\}$
- $\forall p \in \{0, 1, \dots, n\}$ a **p-chain** is a formal sum

$$\alpha = \sum_{i=0}^{n_p} \lambda_i \cdot \sigma_i^{(p)} \qquad \lambda_i \in G, \ \sigma_i \in K$$

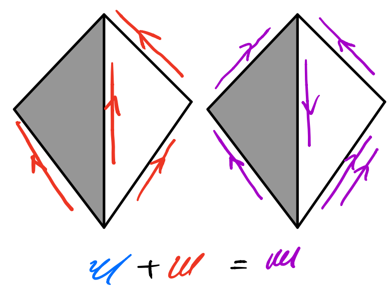
ORIENTED!



$\langle a,d \rangle + 2\langle d,b \rangle + \langle b,c \rangle$

- p-chains can be added & subtracted:

$$\sum_{i=0}^{n_p} \lambda_i \sigma_i + \sum_{j=0}^{n_p} \tilde{\lambda}_j \sigma_j = \sum_{i=0}^{n_p} (\lambda_i + \tilde{\lambda}_i) \sigma_i$$



- $C_p(K; G)$ group of p-chains with coef. G.

$$C_p(K; G) \cong G^{n_p}$$

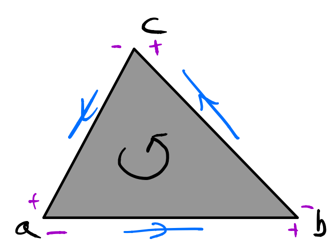
generators / basis: oriented p-sxes.

2. Boundary

- $\sigma = \langle v_0, v_1, \dots, v_p \rangle$ oriented p-sx

boundary of σ $\quad \partial\sigma \in C_{p-1}(K; G)$

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \underbrace{\langle v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p \rangle}_{\text{drop } v_i \text{ from } \sigma}$$



$\partial \langle a,b,c \rangle = \langle a,b \rangle + \langle b,c \rangle + \langle c,a \rangle$

$\partial^2 \langle a,b,c \rangle = 0$

- This induces map (homomorphism / linear map)

$$\partial_p : C_p(K; G) \to C_{p-1}(K; G)$$

$$\sum \lambda_i \sigma_i \mapsto \sum \lambda_i \cdot \partial\sigma_i$$

- chain complex:

$$\cdots \to C_2(K;G) \xrightarrow{\partial_2} C_1(K;G) \xrightarrow{\partial_1} C_0(K;G) \xrightarrow{\partial_0 = 0} 0$$

THM: $\partial^2 = 0 \qquad (\forall p \geq 0: \ \partial_p \circ \partial_{p+1} = 0)$.

Proof:

$\sigma = \langle v_0, v_1, \dots, v_p \rangle$

drop v_i first (∂) — drop v_j later (∂) — ∂^2 — drop v_j first, drop v_i then

$$\underbrace{\left((-1)^i(-1)^{j-1} + (-1)^j(-1)^i\right)}_{0} \langle v_0, \dots, \underbrace{v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots}_{\text{dropped } v_i \,\&\, v_j \text{ from } \sigma}, v_p \rangle \qquad \blacksquare$$

Corollary: $\operatorname{Im} \partial_p \subseteq \operatorname{Ker} \partial_{p-1}$.

3. Homology

$\forall p \in \{0, 1, \dots, n\}$ define two subgroups of $C_p(K; G)$

p-cycles $Z_p(K; G) = \operatorname{Ker} \partial_p$ ← potential repres. of p-holes

p-boundaries $B_p(K; G) = \operatorname{Im} \partial_{p+1}$ ← cycles representing trivial holes

p-homology group

$$H_p(K; G) = Z_p(K; G) / B_p(K; G)$$

← holes

← elements: equivalence classes of p-cycles

THM: $K \simeq L \Rightarrow H_*(K; G) \cong H_*(L; G), \quad \forall G$

Homology is a homotopy invariant (as opposed to Z_p, B_p, C_p)

Homology is of the following form: Betti numbers

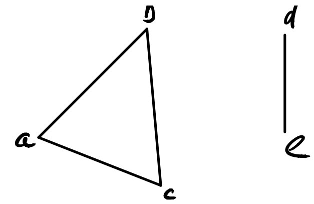
$H_p(K; \mathbb{R}) \cong \mathbb{R}^{D_p}$ $\qquad b_p(K; \mathbb{R}) = D_p$

$H_p(K; \mathbb{Z}_p) \cong \mathbb{Z}_p^{d_p}$ ← p a prime number $\qquad b_p(K; \mathbb{Z}_p) = d_p$

$H_p(K; \mathbb{Z}) \cong \underbrace{\mathbb{Z}^{r_p}}_{\text{free part}} \oplus \underbrace{(\mathbb{Z}_{q_1} \oplus \dots \oplus \mathbb{Z}_{q_k})}_{\text{torsion}}$ $\qquad b_p(K; \mathbb{Z}) = r_p$

ranks of homology groups, "count" the number of p-holes in K.

Example: H_0 with coefficients in G.



$$\begin{array}{ccccc} C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\ \wr\| & & \|\wr & & \\ G^4 & & G^5 & & \end{array}$$

G^5 generated by $\langle a\rangle, \langle b\rangle, \ldots, \langle e\rangle$

G^4 generated by $\langle a,b\rangle, \langle b,c\rangle, \langle a,c\rangle, \langle d,e\rangle$

$$H_0 = \text{Ker}\,\partial_0 / \text{Im}\,\partial_1 \qquad \text{Ker}\,\partial_0 = C_0$$

generators of H_0:

$\partial\langle a,b\rangle = \langle b\rangle - \langle a\rangle$

$[\langle b\rangle] - [\langle a\rangle] = 0$

$[\langle b\rangle] = [\langle a\rangle]$

$$[\langle a\rangle] = [\langle b\rangle] = [\langle c\rangle] \qquad [\langle d\rangle] = [\langle e\rangle]$$

$\partial\langle b,c\rangle = \langle c\rangle - \langle b\rangle$

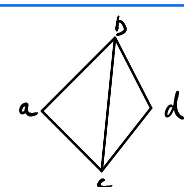
$\partial\langle a,c\rangle = \langle c\rangle - \langle a\rangle$ redundant

$\partial\langle d,e\rangle = \langle e\rangle - \langle d\rangle$

\# of components (0-holes)

Proposition: $H_0(K;G) \cong G^{\#\text{ of components}}$

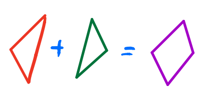
Example: H_1 with coefficients in $\mathbb{Z}_2$



2 holes

$$\begin{array}{ccccc} C_2(K;\mathbb{Z}_2) & \longrightarrow & C_1(K;\mathbb{Z}_2) & \longrightarrow & C_0(K;\mathbb{Z}_2) \\ \| & & \| & & \| \\ 0 & & \mathbb{Z}_2^5 & & \mathbb{Z}_2^4 \end{array}$$

3 non-trivial 1-cycles + 1 trivial 1-cycle



$$H_1(K;\mathbb{Z}_2) = \text{Ker}\,\partial_1 / \text{Im}\,\partial_2 = \text{Ker}\,\partial_1$$

4 elements

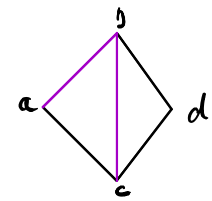
$\mathbb{Z}_2^{2}$

2 holes

Example: H_1 with coefficients in G

K a planar graph.

$$C_2(K;G) \longrightarrow C_1(K;G) \longrightarrow C_0(K;G)$$
$$C_2(K;G) = 0, \quad C_1(K;G) = G^5, \quad C_0(K;G) = G^4$$



$$H_1(K;G) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2 = \operatorname{Ker}\partial_1$$

basis of C_1: $\langle a,b\rangle$ $\langle b,d\rangle$ $\langle d,c\rangle$ $\langle a,c\rangle$ $\langle b,c\rangle$

boundaries: $\langle b\rangle - \langle a\rangle$ (obsolete), $\langle d\rangle - \langle b\rangle$, $\langle c\rangle - \langle d\rangle$, $\langle c\rangle - \langle a\rangle$, $\langle c\rangle - \langle b\rangle$ (obsolete)

$$\langle c\rangle - \langle b\rangle = \underbrace{\langle c\rangle - \langle d\rangle}_{3^{rd}} + \underbrace{\langle d\rangle - \langle b\rangle}_{2^{nd}}$$

$$\langle b\rangle - \langle a\rangle = -\underbrace{(\langle d\rangle - \langle b\rangle)}_{2^{nd}} - \underbrace{(\langle c\rangle - \langle d\rangle)}_{3^{rd}} + \underbrace{\langle c\rangle - \langle a\rangle}_{4^{th}}$$

"$\operatorname{Ker}\partial_1$" is generated by 2 elements $\rightsquigarrow$ 2 holes

Proposition: K a planar graph $\Rightarrow$ $H_1(K;G) \cong G^{\#\text{ of holes}}$

Computing homology with matrix reductions

choose a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{Z}_p \text{ for } p \text{ prime}\}$

K a scx of dim n. Fix $q \in \{0,1,\dots,n\}$

$$H_q(K;\mathbb{F}) \cong \mathbb{F}^{r_q} \cong \operatorname{Ker}\partial_q / \operatorname{Im}\partial_{q+1}$$

Proposition: $f: A \to B$ linear map of vector spaces, $\operatorname{rank} f = \dim f(A)$. Then:

(i) $\dim A = \dim \operatorname{Ker} f + \operatorname{rank} f$

(ii) $\dim(B/f(A)) = \dim B - \operatorname{rank} f$

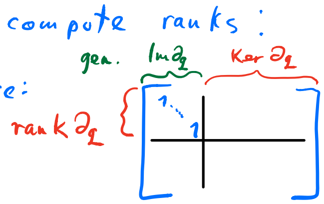
Corollary: ⓐ $\dim \operatorname{Ker}\partial_q = \dim C_q - \operatorname{rank}\partial_q =$
$= (\#\text{ of } q\text{-sxes}) - \operatorname{rank}\partial_q$

ⓑ $r_q = (\#\text{ of } q\text{-sxes}) - \operatorname{rank}\partial_q - \operatorname{rank}\partial_{q+1}$.

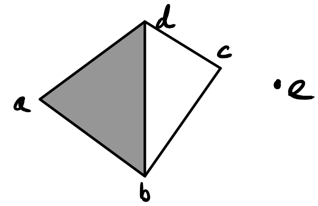
How do we compute ranks: $\partial_q \rightsquigarrow$ Matrix $\rightsquigarrow$ rank

Gauss, ...

The nicest case: SMITH NORMAL FORM



Example: H_* with coef. in a field $\mathbb{F}$



$$\underset{\overset{\shortparallel}{0}}{C_3} \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

- rank $\partial_0 = 0$

- ∂_1

$$\begin{array}{c|ccccc} & \langle a,b\rangle & \langle b,c\rangle & \langle c,d\rangle & \langle a,d\rangle & \langle b,d\rangle \\ \hline \langle a\rangle & -1 & & & -1 & \\ \langle b\rangle & 1 & -1 & & & -1 \\ \langle c\rangle & & 1 & -1 & & \\ \langle d\rangle & & & 1 & 1 & 1 \\ \langle e\rangle & & & & & \end{array}$$

$\rightsquigarrow$ Gauss. through rows

$$\begin{bmatrix} -1 & & & -1 & \\ & -1 & & -1 & -1 \\ & & -1 & -1 & -1 \\ & & & & \\ & & & & \end{bmatrix} \rightsquigarrow \text{rank } \partial_1 = 3$$

- ∂_2

$$\begin{array}{c|c} & \langle a,b,d\rangle \\ \hline \langle a,b\rangle & 1 \\ \langle b,c\rangle & 0 \\ \langle c,d\rangle & 0 \\ \langle a,d\rangle & -1 \\ \langle b,d\rangle & 1 \end{array}$$

rank $\partial_2 = 1$

- ∂_3 trivial

Conclusion: $r_0 = 5 - 0 - 3 = 2$ components

$r_1 = 5 - 3 - 1 = 1$ hole

$r_2 = 1 - 1 - 0 = 0$ caves / voids

How to get representatives? TO CONTINUE NEXT WEEK

$$\begin{array}{ccccc} \langle a,b\rangle & \langle b,c\rangle & \langle c,d\rangle & \langle a,d\rangle & \langle b,d\rangle \\ \end{array}$$

$$\begin{bmatrix} -1 & & & -1 & \\ & -1 & & -1 & -1 \\ & & -1 & -1 & -1 \\ & & & & \\ & & & & \end{bmatrix} \rightsquigarrow \begin{array}{ccc|} \langle a,b\rangle & \langle b,c\rangle & \langle c,d\rangle \\ -1 & & \\ & -1 & \\ & & -1 \\ & & \\ & & \end{array}$$