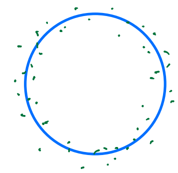
Constructions of complexes

Setting: X a metric space (typically $\mathbb{R}^n$)

$S \subset X$ a finite subset (possibly representing some shape)



S

TASK: Model the shape by some scx on S.

Def: Rips cx (Vietoris-Rips complex)

choose a scale $r > 0$

$\mathrm{Rips}(S, r)$ is a scx:

① Vertices $= S$

② $\sigma \in \mathrm{Rips}(S,r) \Leftrightarrow \mathrm{diam}\, \sigma \leq r$.

Properties:
- easy to construct
- Rips is an ascx, typically not embeddable into X.
- $\mathrm{Rips}(S,r)$ is a discrete set for small r.
- $\mathrm{Rips}(S,r) = \Delta^{|S|-1}$ for large r.
- $r_1 < r_2 \Rightarrow \mathrm{Rips}(S, r_1) \hookrightarrow \mathrm{Rips}(S, r_2)$
- Rips filtration $\{\mathrm{Rips}(S,r)\}_{r>0}$
- Rips complex is a special case of Clique cx.

G a graph

$\mathrm{Clique}(G)$ is ascx:

$\{v_0, v_1, \dots, v_k\} \in \mathrm{Clique}(G) \Leftrightarrow \forall i,j : v_i v_j \in G$

Def: Čech cx

Choose scale $r > 0$

$B(x,r) = \{y \in X ; d(x,y) \leq r\}$

$\check{C}\mathrm{ech}(S,r)$ is an ascx:

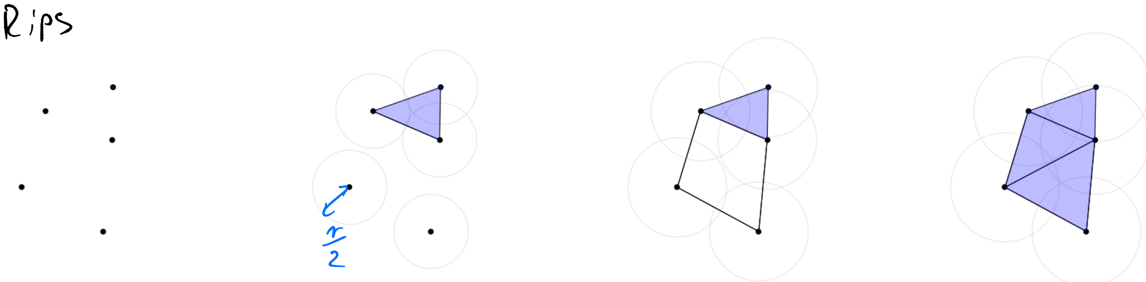
ⓐ Vertices .... points of S

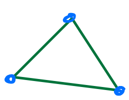
ⓑ $\sigma \in \check{C}\mathrm{ech}(S,r) \Leftrightarrow \bigcap_{x \in \sigma} B(x,r) \neq \emptyset \Leftrightarrow \exists y \in X : B(y,r) \supseteq \sigma$

Properties:

- ~~easy~~ to construct
- Čech is an ascx, typically not embeddable into X.
- $\check{C}ech(S,r)$ is a discrete set for small r.
- $\check{C}ech(S,r) = \Delta^{|S|-1}$ for large r.
- $r_1 < r_2 \Rightarrow \check{C}ech(S,r_1) \hookrightarrow \check{C}ech(S,r_2)$
- Čech filtration $\{\check{C}ech(S,r)\}_{r>0}$

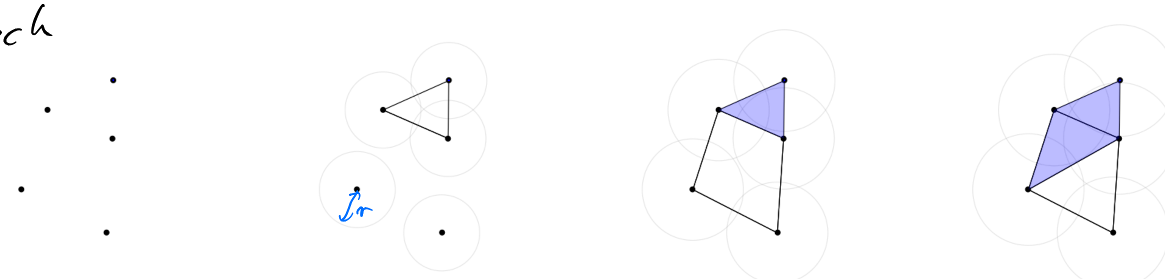
Rips





does not happen in Rips

Čech

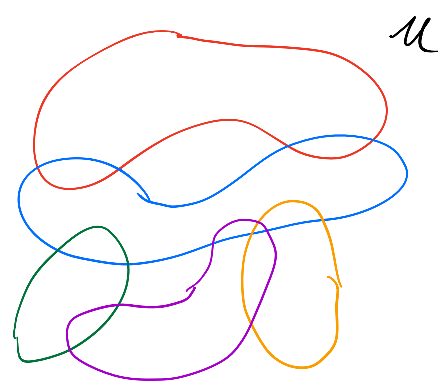


Def: Let $\mathcal{U}$ be a collection of subsets of X.

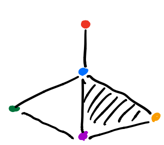
Nerve $\mathcal{N}(\mathcal{U})$ is an ascx:

a) Vertices ... elements of $\mathcal{U}$

b) $\sigma \in \mathcal{N}(\mathcal{U}) \Leftrightarrow \bigcap_{U \in \sigma} U \neq \emptyset$.



$\mathcal{N}(\mathcal{U})$



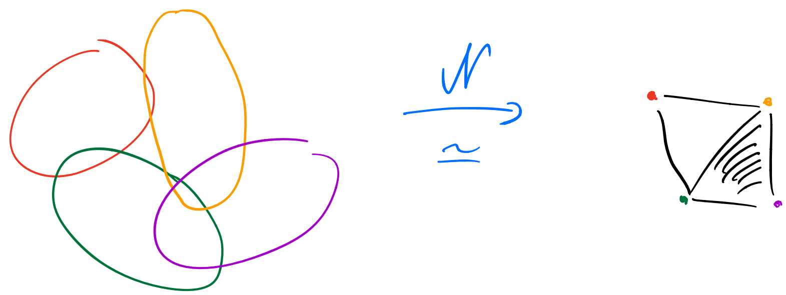
$\check{C}ech(S,r) = \mathcal{N}(\{B(x,r)\}_{x\in S})$

Nerve THM: $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$ consists of closed convex sets in $\mathbb{R}^d$.

Then $\mathcal{N}(\mathcal{U}) \simeq \bigcup_{i=1}^{k} U_i$.

Corollary: $\check{C}ech(S, r) \simeq \bigcup_{x \in S} B(x, r)$.

Example:



A connection between Rips & Čech:

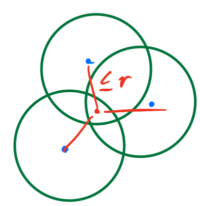
→ If $X = \mathbb{R}^d \Rightarrow Rips^{(1)}(X, r) = \check{C}ech^{(1)}(X, \frac{r}{2})$

→ $Rips(S, 2r) \supseteq \check{C}ech(S, r)$

$Rips(S, r) \subseteq \check{C}ech(S, r)$

for $X = \mathbb{R}^d$ [Jung's THM]

$Rips(S, r\sqrt{2}) \subseteq \check{C}ech(S, r)$



A few more examples of Nerves:

① Delaunay triangulation is the nerve of the Voronoi decomposition.

② α-complexes:

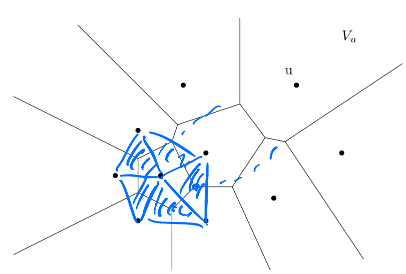
$S = \{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$

$\forall i$: V_i = the Voronoi region of v_i

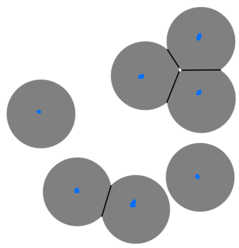
Define $\forall i$: $A_i = V_i \cap B(v_i, r)$

α-cx @ $r > 0$: $\mathcal{N}(A_i)$

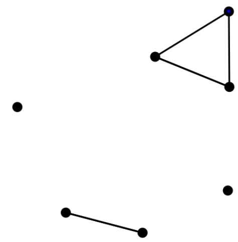
- Representable in $\mathbb{R}^n$
- models molecules, proteins, etc.

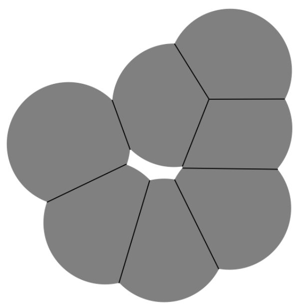


α - shapes $\qquad\qquad$ α - cx

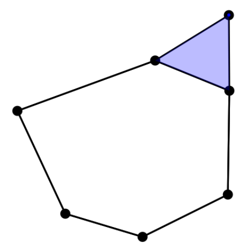


$\simeq$





$\simeq$



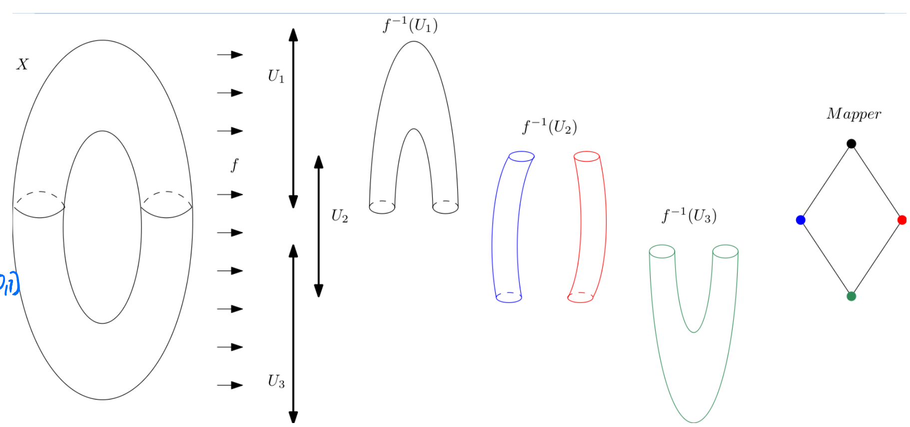
③ Mapper

Theoretically:

X .... space

$f : X \to [0,1]$

$\mathcal{U} = \{U_i\}$ cover of $[0,1]$



Mapper is a graph:

→ vertices .... components of $f^{-1}(U_i)$

→ edge $A - B \iff A \cap B \neq \emptyset$

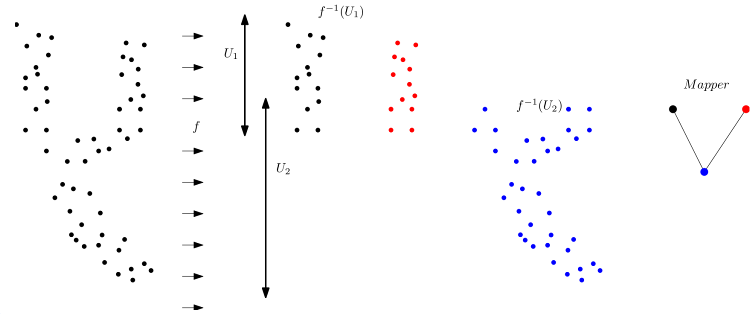
Practically:

X is a point cloud

$f : X \to I$ a measurment

$\mathcal{U}$ ... partition of I

choose a clustering scheme



Mapper is a graph:

→ vertices : clusters of $f^{-1}(U_i)$

→ edges : as before

Computing Čech complexes.

Constructing Čech(S, r) :

→ $\forall \sigma \subset S \subset \mathbb{R}^n$ compute the minimal ball $B(y, \delta)$ containing σ

→ if $\delta \leq r$, include σ into Čech(S, r)

Miniball algorithm:

Input : disjoint sets τ, ν

Output : minimal ball with:

→ ν on the boundary

→ τ in the ball

works for appropriate sets ν & τ

Miniball (τ, ν)

If $\tau = \emptyset$ compute miniball directly /* In this case $|\nu| \leq n+1$ */

else choose $u \in \tau$

$B = \text{miniball}(\tau \setminus \{u\}, \nu)$

if $u \notin B$, $B = \text{miniball}(\tau \setminus \{u\}, \nu \cup \{u\})$

return B

To get miniball(σ) call Miniball$(\sigma, \emptyset)$

Interleavings

① Rips / Čech $\quad \check{C}ech(S,r) \supseteq Rips(S,r) \supseteq \check{C}ech(S,\frac{r}{2})$

$$\begin{tikzcd}
\cdots \arrow[r, hook] & Rips(S,r) \arrow[r, hook] \arrow[d] & Rips(S,2r) \arrow[r, hook] \arrow[d] & \cdots \\
\cdots \arrow[r, hook] & \check{C}ech(S,r) \arrow[r, hook] \arrow[ur] & \check{C}ech(S,2r) \arrow[r, hook] \arrow[ur] & \cdots
\end{tikzcd}$$

filtration

interleaving

filtration

Interleaving: a collection of maps between two filtrations that commute with the bonding maps.

② Perturbation interleaving

$S = \{v_1, v_2, \dots, v_k\}$ sample $\varepsilon > 0$

perturb

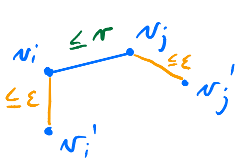
$S' = \{v_1', v_2', \dots, v_k'\}$ $\forall i: d(v_i, v_i') \le \varepsilon$

Problem: at each $r > 0$ $Rips(S,r)$ & $Rips(S',r)$ can be very different

Relief: filtrations are very similar.

2ε-interleaving

$$\begin{tikzcd}
\cdots \arrow[r] & Rips(S,r) \arrow[r, hook] \arrow[dr] & Rips(S,r+2\varepsilon) \arrow[r, hook] \arrow[dr] & Rips(S,r+4\varepsilon) \arrow[r] & \cdots \\
\cdots \arrow[r] & Rips(S',r) \arrow[r, hook] \arrow[ur] & Rips(S',r+2\varepsilon) \arrow[r, hook] \arrow[ur] & Rips(S',r+4\varepsilon) \arrow[r] & \cdots
\end{tikzcd}$$



$\langle v_i, v_j \rangle \in Rips(S,r)$

$\Rightarrow d(v_i', v_j') \le r + 2\varepsilon \Rightarrow \langle v_i', v_j' \rangle \in Rips(S', r+2\varepsilon)$

Def: Filtrations $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_i\}$ are δ-interleaved for $\delta > 0$ if

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, hook] \arrow[dr] & A_{i+\delta} \arrow[r, hook] \arrow[dr] & A_{i+\delta} \arrow[r] & \cdots \\
\cdots \arrow[r] & B_i \arrow[r, hook] \arrow[ur] & B_{i+\delta} \arrow[r, hook] \arrow[ur] & B_{i+2\delta} \arrow[r] & \cdots
\end{tikzcd}$$

HW: Are Čech filtrations of S & S' also interleaved & what is the parameter?

Groups

A group is a collection of elements with one invertible operation.

Def: A group $(A,+)$ is a set with an associative binary operation $+: A\times A \to A$, such that: $(a+b)+c = a+(b+c)$

a) $\exists\, 0\in A: \ 0+a = a+0 = a, \ \forall a\in A$

b) $\forall a\in A \ \exists\, -a\in A: \ \underbrace{a+(-a)}_{a-a} = 0.$

A group is Abelian (commutative) if: $\forall a,b\in A: \ a+b = b+a.$

! All our groups will be abelian.

Example: • $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{C},+)$, $(\mathbb{Q}\setminus\{0\},\cdot)$, ...

• $(\mathbb{R}^n,+)$

• (rotations of S^1, composition)

• (functions: $D\to\mathbb{R}$, pointwise $+$)

• $q\in\mathbb{N}$

$\mathbb{Z}_q = \{0,1,\dots,q-1\}$ remainders after division by q.

operation: $a+b \pmod q$ addition mod q.

Example: • in $\mathbb{Z}_5$:

$1+1=2$ $\quad$ $3+4=2$

$2+3=0 \rightsquigarrow -3=2$

• in $\mathbb{Z}_2$: $0+0=0$ $\quad$ $1+1=0$ $\quad$ $a+b = a \text{ XOR } b$

$0+1=1$ $\quad$ $1+0=1$ $\quad$ exclusive or

We can also multiply in $\mathbb{Z}_q$: $\pmod q$

Example: • In $\mathbb{Z}_5$: $2 \cdot 3 = 1$ $\quad 2 \cdot 4 = 3$ $\quad 4 \cdot 4 = 1$

$3 \cdot 3 = 4$ $\quad 0 \cdot 3 = 0$

• In $\mathbb{Z}_2$: $0 \cdot 0 = 0$
$0 \cdot 1 = 0$
$1 \cdot 1 = 1$

$a \cdot b = a \wedge b$

Can we also divide (except by 0):

$$\frac{a}{b} = a \cdot b^{-1}$$

$b \cdot b^{-1} = 1$

Example: $\mathbb{Z}_5$ $\quad 2^{-1} = 3$
$3^{-1} = 2$
$4^{-1} = 4$
$1^{-1} = 1$

If all nonzero elements in $\mathbb{Z}_q$ have an inverse, $\mathbb{Z}_q$ is a field.

Example: $\mathbb{Z}_5$ is a field

$\mathbb{Z}_4$ is not a field: $2 \cdot 2 = 0$
$2 \cdot 3 = 2$
$3 \cdot 3 = 1$

2 does not have an inverse.

$\mathbb{Z}_q$ is a field iff q is a prime.