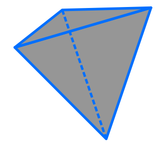
Simplicial complexes [scx]



① Building block: a geometric simplex [gsx] in $\mathbb{R}^d$

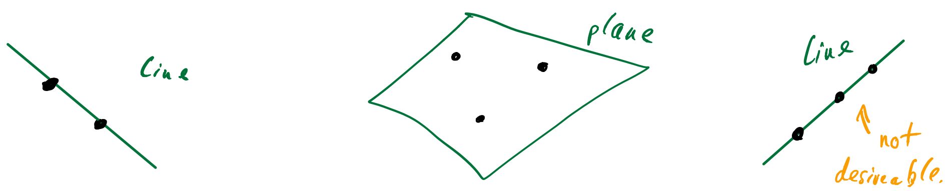
$V = \{v_0, v_1, \dots, v_k\} \subset \mathbb{R}^d$

Def: an affine combination of V is any expression (point) of the following f.:

$$\sum_{i=0}^{k} \alpha_i \cdot v_i \,, \quad \sum_{i=0}^{k} \alpha_i = 1.$$

(if $\alpha_i \geq 0, \forall i \rightsquigarrow$ convex combination).

Example



Def: V are affinely independent if no v_i is an affine combination of other points in V.

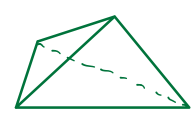
Prop: V are aff. independent $\iff \{v_1 - v_0, v_2 - v_0, \dots, v_k - v_0\}$ is lin. indep.

Example: 2 pts are aff. indep iff they are different

3 - - - - - - - - iff they are not colinear

4 - - - - - - - - - iff they do not lie on the same plane

Def: A geometric simplex [sx] $\sigma = \sigma^k$ in $\mathbb{R}^d$ is $\mathrm{conv}(V)$ of an affinely independent set V.



→ V ... vertices of σ
→ $\dim(\sigma) = k$
→ edges of σ: convex combos of pairs from V.

→ σ is spanned by V, or $\sigma = \langle v_0, v_1, \dots, v_k \rangle$

→ if τ is a sx spanned by a subset of V, we say:

- τ is a face of σ $\quad \tau \leq \sigma$
- σ is a coface of τ
- τ is a facet of σ if $\dim(\tau) = \dim(\sigma) - 1$.

→ $\sigma \cong D^K$

→ Each $x \in \sigma$ can be expressed uniquely as

$$x = \sum_{i=0}^{K} \underbrace{\alpha_i}_{} v_i \longrightarrow \text{barycentric coordinates of } x.$$

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② Geometric simplicial complexes [gscx, scx]

Def: A gscx K in $\mathbb{R}^d$ is a (finite) collection of geometric simplices, so that:

① $\sigma \in K,\ \tau \leq \sigma \Rightarrow \tau \in K$

② $\sigma, \tau \in K \Rightarrow \sigma \cap \tau$ is a face of σ and τ (or empty).

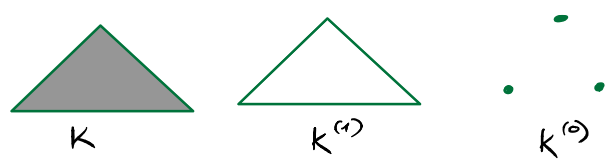
↳ $\dim(K) = \max_{\sigma \in K} \dim(\sigma)$

↳ $\mathrm{vertices}(K) = \bigcup_{\sigma \in K} \mathrm{vertices}(\sigma)$

↳ $\mathrm{edges}(K) = \bigcup_{\sigma \in K} \mathrm{edges}(\sigma)$

Def: A geometric scx L is a subcomplex of a gscx K if $L \subseteq K$ $[L \leq K]$.

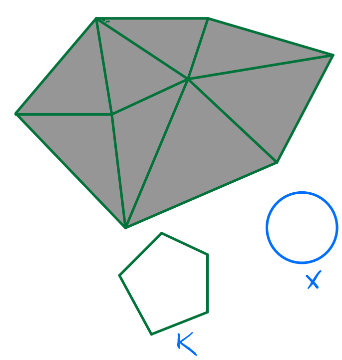
↳ $n \in \mathbb{N}$, n-skeleton of K : $K^{(n)} = \{\sigma \in K \,;\, \dim(\sigma) \leq n\}$



↳ $|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^d$ body of a scx K.

Def: A triangulation on $X \subset \mathbb{R}^d$ is a gscx K: $|K| \cong X$.

gscx L is a subdivision of a gscx K if it is obtained from K by subdividing sxes. $(|L| = |K|)$.



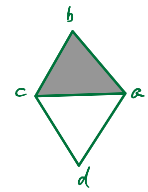
③ Abstract simplicial complexes [ascx]

idea: forget the coordinates of vertices in a gscx, assign labels instead.

Def: Let W be a finite set. An ascx $\mathcal{K}$ on W is a family of subsets of W satisfying:

$$F \in \mathcal{K},\ G \subset F \Rightarrow G \in \mathcal{K}.$$

Example:



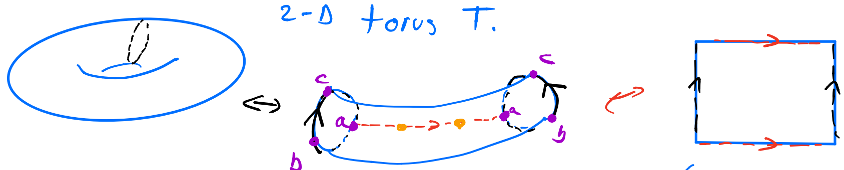
$\mathcal{K} = \{abc, ad, dc, ab, bc, ac, a, b, c, d\}$

↳ abstract sx [asx]: any non-empty element of $\mathcal{K}$

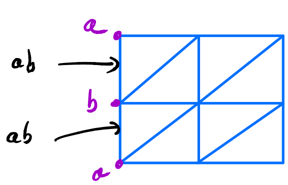
↳ $F \subset G \in \mathcal{K}$:

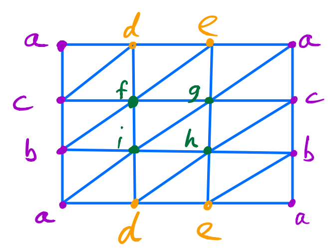
- F is a face of G
- G is a coface of F
- $\dim(F) = |F| - 1$.
- $\dim \mathcal{K} = \max_{H \in \mathcal{K}} \dim(H)$

Example: 2-D torus T.



triangulation





acd, def, ...

18-triangles

$\#_1$ - edges

$\#_2$ - vertices

$$\text{gscx } K \xrightarrow{\text{forget the coord's, assign labels}} \text{ascx } \mathcal{K}$$

$$\text{gscx } K \xleftarrow{\text{think of coordinates for each label}} \text{ascx } \mathcal{K}$$

(construct a geometric realization of $\mathcal{K}$)

Thm: Every ascx $\mathcal{K}$ of dimension d admits a geometric realization in $\mathbb{R}^{2d+1}$.

@ $d=1$, $\mathcal{K}$ is a graph. There are graphs, which are not planar. K_5

Def: A topological invariant $i(X)$ is an object (#, vector, function, ...) assigned to a metric space so that: $X \cong Y \Rightarrow i(X) = i(Y)$.

(Similarly: homotopy invariants: $X \simeq Y \Rightarrow j(X) = j(Y)$).

Two homotopy invariants:

1. # of components
2. χ Euler characteristic of an scx K.

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \cdot [\#\text{ of } i\text{-sxes of } K].$$

↳ $\chi(D^n) = 1$, for $n \in \mathbb{N}$.

↳ $\chi(\text{Torus}) = 0$.

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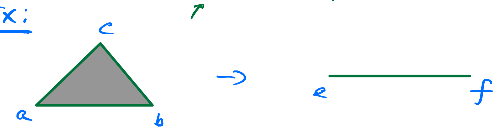
4 Simplicial maps

Def: Assume K, L are ascx's. A simplicial map $f: K \to L$ is an assignment $f': K^{(0)} \to L^0$ satisfying

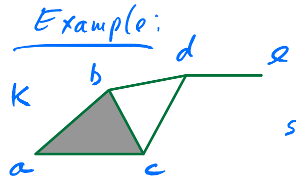
$$\sigma = \{v_0, v_1, \dots, v_m\} \in K \Rightarrow f'(\sigma) = \{\underbrace{f'(v_0), f'(v_1), \dots, f'(v_m)}_{\text{we could see repetition here}}\} \in L \qquad (*)$$

Ex:



$a \mapsto e$
$b \mapsto e$
$c \mapsto f$

Example:



simplicial map

$g: K \to K$

$a \mapsto a$
$b \mapsto c$
$c \mapsto c$
$d \mapsto d$
$e \mapsto b$

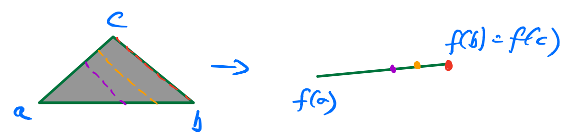
Def: Assume K, L are gscx's. Map $f: K \to L$ is *simplicial* if:

ⓐ the induced map on vertices is simplicial

(f(vertex) = vertex & (*))

ⓑ f is linear on each sx $\{v_0, v_1, \dots, v_k\}$:

$$f\left(\sum_{i=1}^{m} \alpha_i v_i\right) = \sum_{i=1}^{m} \alpha_i f(v_i) \qquad \forall \alpha_i \in [0,1], \ \sum_{i=1}^{m} \alpha_i = 1$$



Proposition: K, L gscx's.

① Each simplicial map is continuous.

② for each continuous map $g: K \to L$ there exist $n \in \mathbb{N}$ and a simplicial map $f: \tilde{K} \to \tilde{L}$, such that $f \simeq g$.

n^{th} barycentric subdivisions of K & L.

Example: