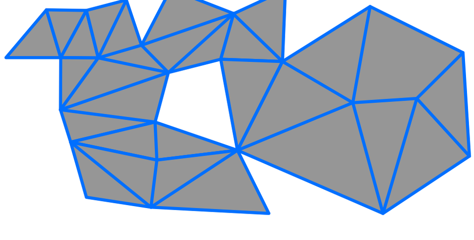
Triangulations in the plane

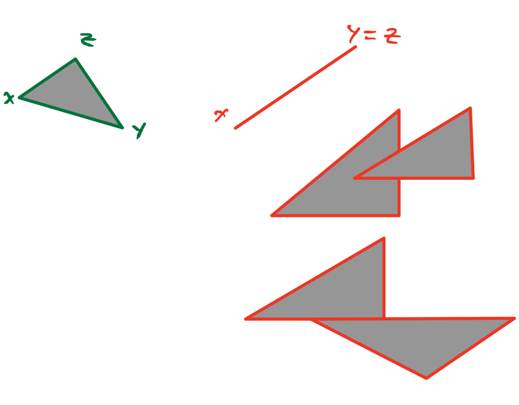


Def: A triangulation of $D \subset \mathbb{R}^2$ is a decomposition of D into triangles, so that:

ⓐ No triangle is degenerate

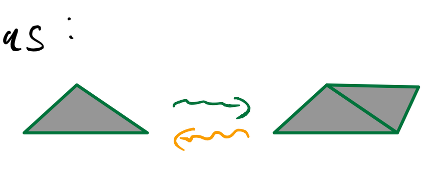
ⓑ The interiors of triangles disjoint

ⓒ Intersections of triangles are either a common point, edge or empty



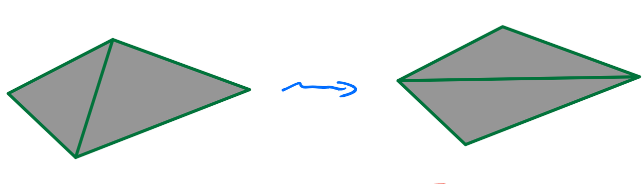
Modifications of triangulations:

ⓐ Add a triangle

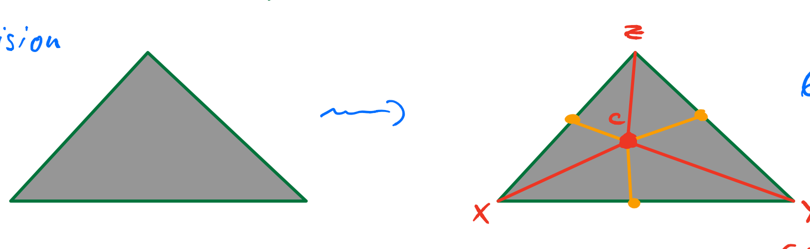


ⓑ Remove a triangle

ⓒ Flipping a common edge



ⓓ Barycentric subdivision



6 triangles

$c = \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z$

General task: Given $S = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}$, find a triangulation on S

vertices .... S

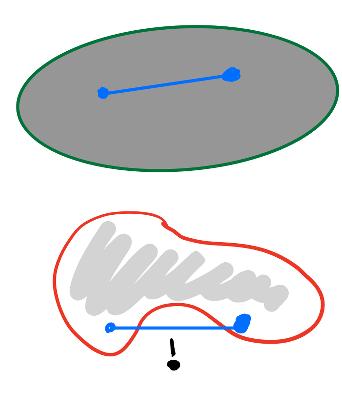
triangulate $\mathrm{Conv}(S)$.

Recap on convexity:

- $C \subset \mathbb{R}^n$ is convex, if $\forall x, y \in C$, $\forall t \in [0,1]$:

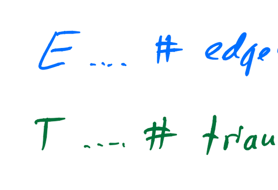
  $tx + (1-t)y \in C$

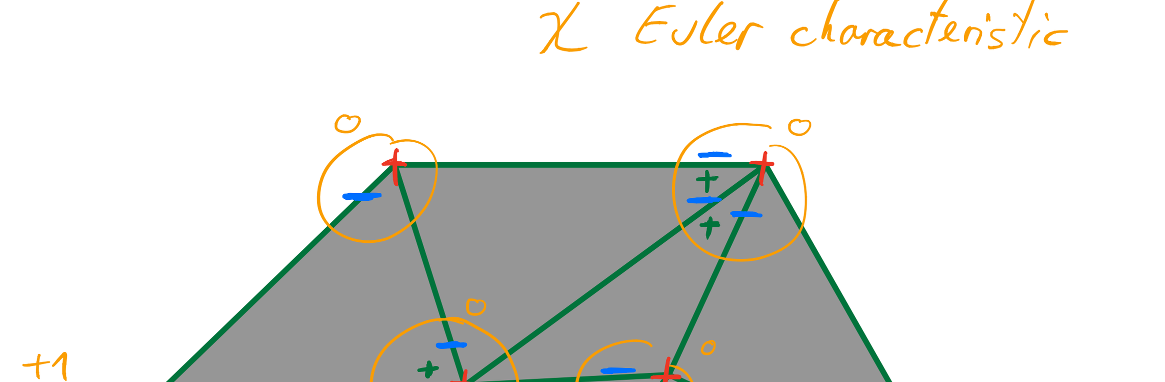
  Parameterization of the line segment $x \to y$



- The convex hull $\mathrm{Conv}(S)$ is the smallest convex set containing S.

$$\mathrm{Conv}(S) = \Big\{ \underbrace{\sum_i \alpha_i \cdot x_i}_{\text{convex combination}} \; ; \; \alpha_i \in [0,1], \sum_i \alpha_i = 1 \Big\}$$





V .... # vertices

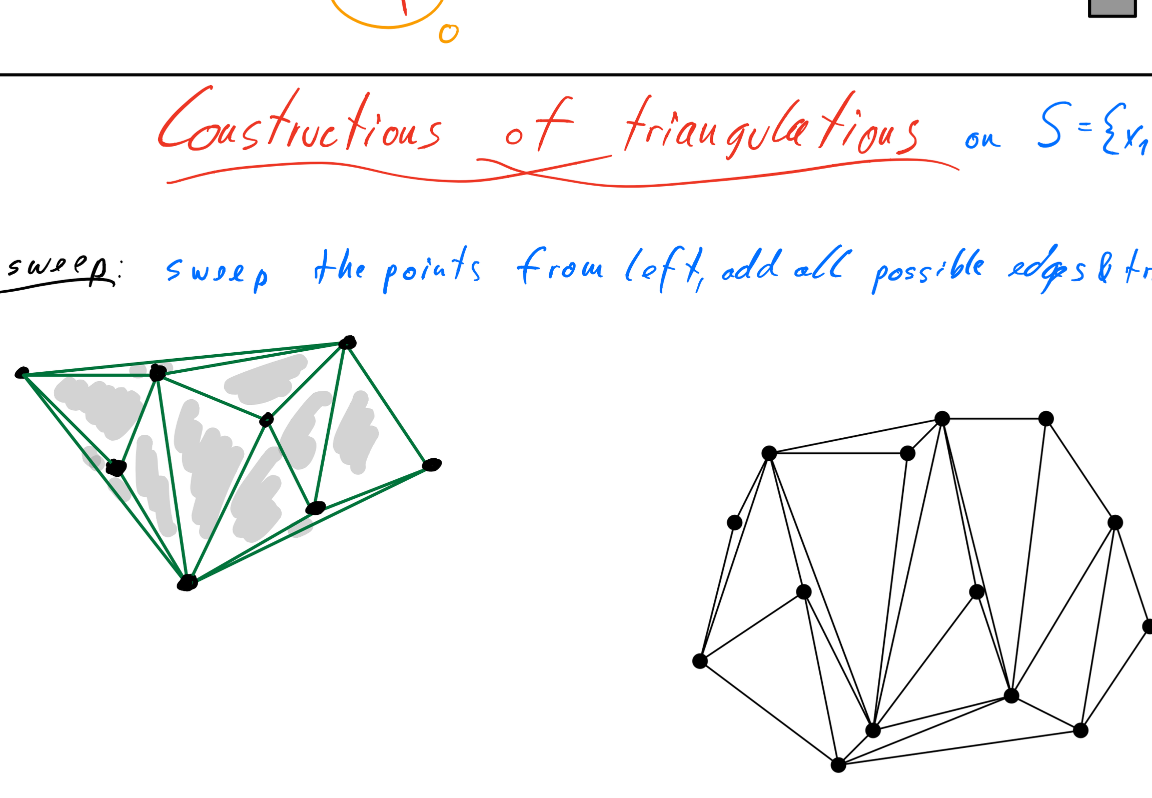
E .... # edges

T .... # triangles

Euler–Poincaré formula: If our region is convex (or $\simeq \bullet$) then

$$\underbrace{V - E + T}_{\chi \text{ Euler characteristic}} = 1$$

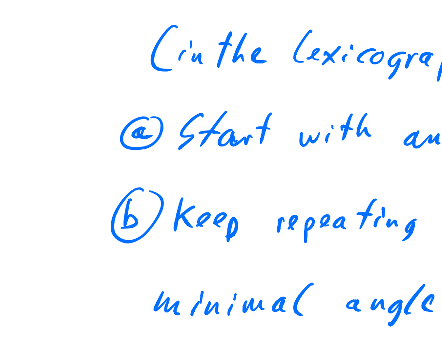
Proof:

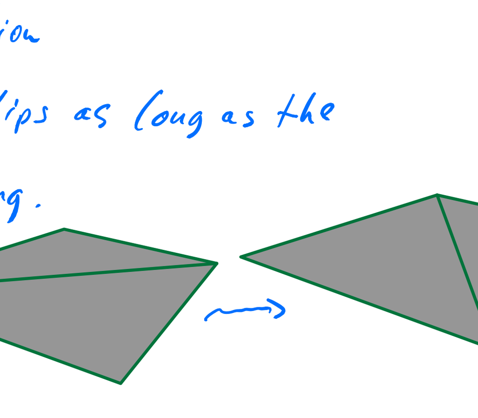


$\chi = 1$

Constructions of triangulations on $S = \{x_1, \dots, x_n\} \subset \mathbb{R}^2$

① Line sweep: sweep the points from left, add all possible edges & triangles.

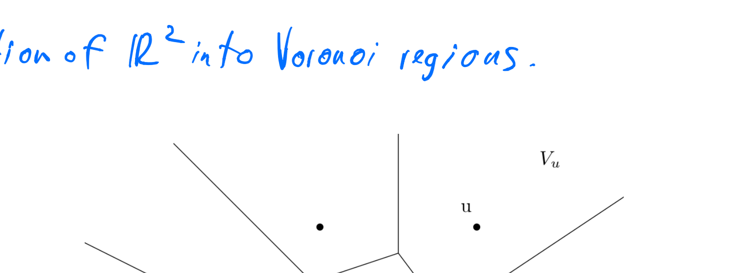




② Max Min angle: A triangulation maximizing (the vector of) minimal angles (in the lexicographical order).

ⓐ Start with any triangulation

ⓑ Keep repeating the edge flips as long as the minimal angle is increasing.

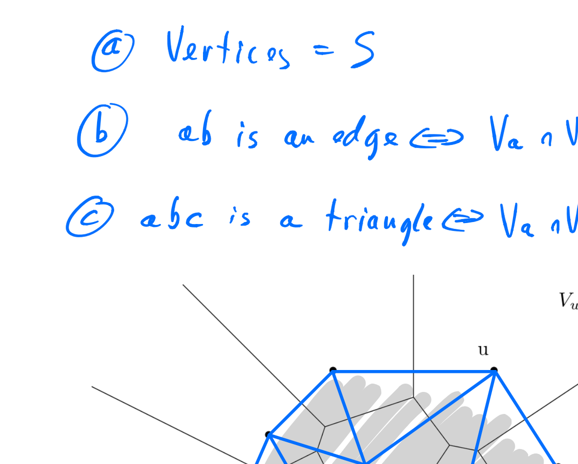


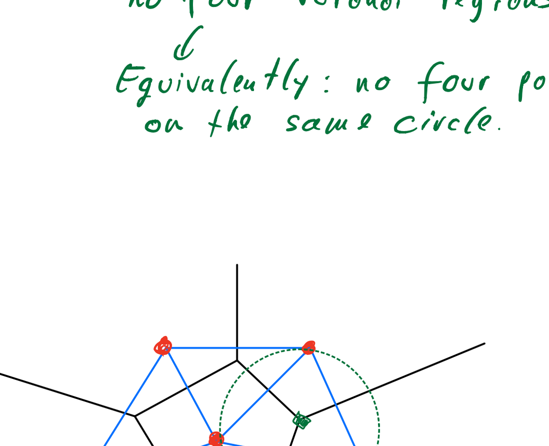
Voronoi diagram & Delaunay triangulation of $S \overset{\text{FIN}}{\subset} \mathbb{R}^2$

Def: The Voronoi region V_a of $a \in S$:

$$V_a = \{x \in \mathbb{R}^2;\ d(x,a) \leq d(x,a') \quad \forall a' \in S\}$$

The Voronoi diagram is a decomposition of $\mathbb{R}^2$ into Voronoi regions.





Def: The Delaunay triangulation on S is the dual of the Voronoi diagram:

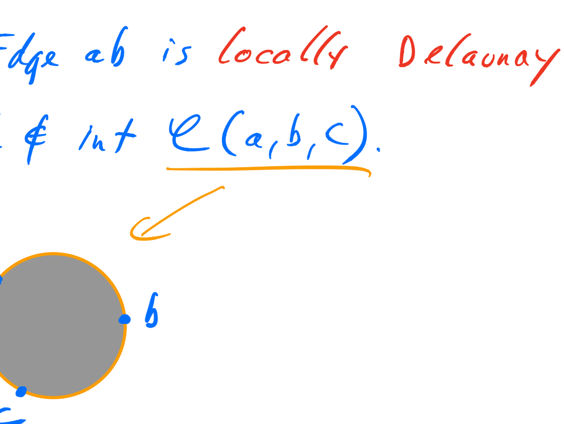
ⓐ Vertices $= S$

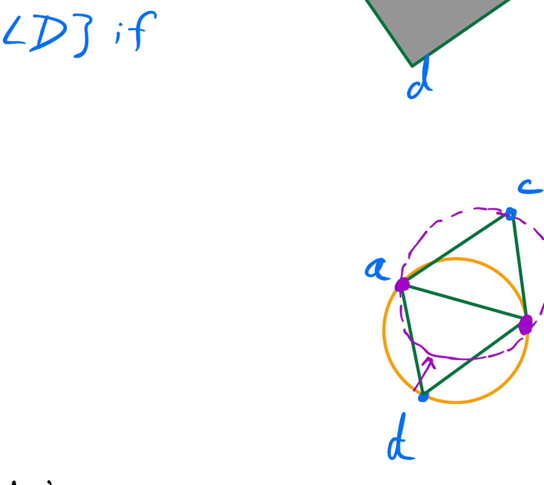
ⓑ ab is an edge $\Leftrightarrow V_a \cap V_b \neq \emptyset$

ⓒ abc is a triangle $\Leftrightarrow V_a \cap V_b \cap V_c \neq \emptyset$

← Additional assumption on S: no four Voronoi regions intersect

Equivalently: no four points lie on the same circle.





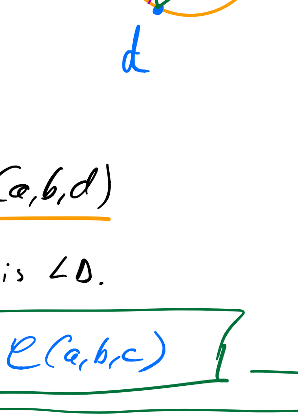
S

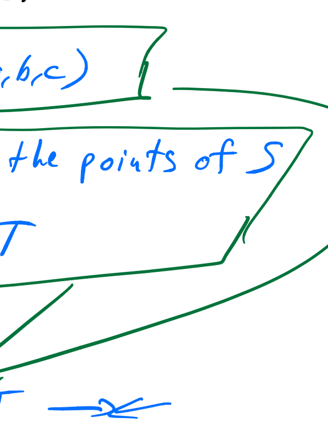
Voronoi decomposition

Delaunay triangulation

$\mathcal{C}$

Def: let $T(S)$ be a triangulation on S. Suppose edge ab is shared in $T(S)$ by abc & abd. Edge ab is locally Delaunay [LD] if $d \notin \mathrm{int}\, \mathcal{C}(a,b,c)$.





Proposition: ① Def. of LD is symmetric:

$$d \notin \mathrm{int}\, \mathcal{C}(a,b,c) \Leftrightarrow c \notin \mathrm{int}\, \mathcal{C}(a,b,d)$$

② Each edge in a Delaunay triangulation is LD.

Proof: ② By contradiction: assume $d \in \mathrm{int}\, \mathcal{C}(a,b,c)$

$T = V_a \cap V_b \cap V_c \Rightarrow \{a,b,c\}$ are the points of S closest to T

but d is closer to T ↯

Hence $d \notin \mathrm{int}\, \mathcal{C}(a,b,c)$. ∎

THM: A triangulation is Delaunay iff each edge is LD.

Obvious algorithm: ⓐ Start with any triangulation

ⓑ Keep flipping non-LD edges.

Nice ..... but:

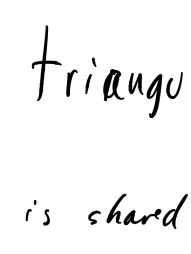
(i) How do we know whether an edge is LD? ✓

(ii) Does it terminate? ✓ @ the Delaunay triangulation.

✓ **Incircle test:** $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$, $d = (d_1, d_2)$

Assume (a,b,c) is positively oriented:

pic



formula

$$\begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix} > 0$$

Then

$d \in \mathcal{C}(a,b,c)$ iff

$$\begin{vmatrix} a_1 & a_2 & a_1^2 + a_2^2 & 1 \\ b_1 & b_2 & b_1^2 + b_2^2 & 1 \\ c_1 & c_2 & c_1^2 + c_2^2 & 1 \\ d_1 & d_2 & d_1^2 + d_2^2 & 1 \end{vmatrix} > 0.$$

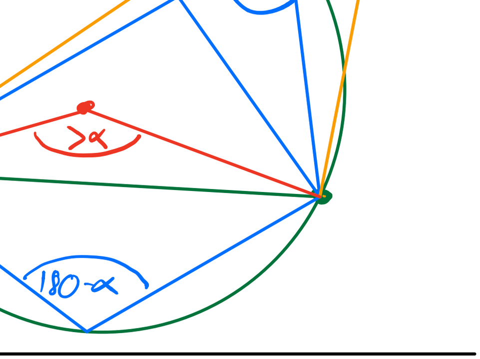
Theorem: The Delaunay triangulation is the MaxMin angle triang.

Proposition: Assume edge ab is shared by abc & abd in a triangulation. Then the following are equivalent:

1) ab is LD $(d \notin \mathcal{C}(a,b,c))$

2) ab is a MaxMin angle

3) $\measuredangle acb + \measuredangle adb < 180°$



Max Min choice

Proof: $1) \overset{\text{IAT}}{\Longleftrightarrow} 3)$

$1) \Leftrightarrow 2)$ ✓

Min Angle > Min Angle

$\delta > \gamma$

⋮

and similarly for the other three cases.

IAT

Inscribed angle theorem [Euclid, OLD]