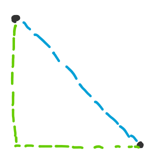
Metric spaces

Tuesday, 6 October 2020 12:33



Def. A pair (X,d) is a **metric space** if $d: X \times X \to [0,\infty)$ satisfying:
1. $d(x,y) = d(y,x)$ symmetry
2. $d(x,y) = 0$ iff $x = y$
3. $d(x,z) \leq d(x,y) + d(y,z)$ triangle inequality

Examples:

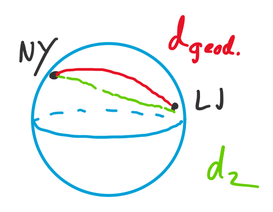
① $X = \mathbb{R}^n$, $n \geq 1$. $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

(i) Euclidean $d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$

(ii) $d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$, $p \geq 1$.

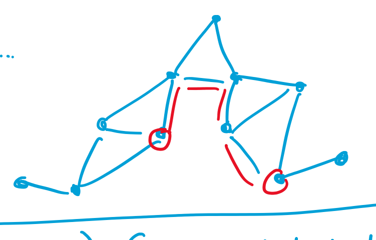
$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

(iii) $d_\infty(x,y) = \max_i \{|x_i - y_i|\}$

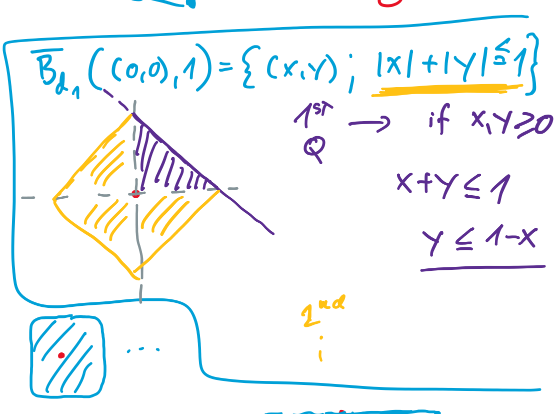


② Geodesic distance... the distance is the length of the shortest path.

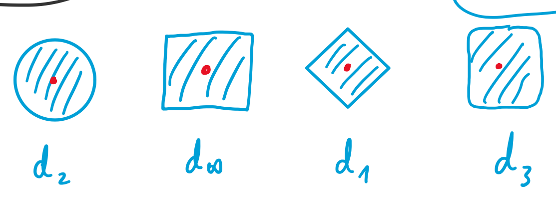
Natural setting: graphs, surfaces, ...

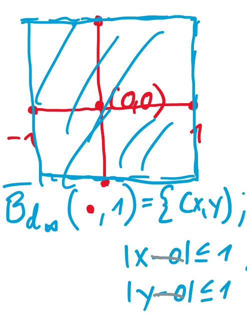


Def. Let (X,d) be a metric space, $x \in X, r > 0, A \subset X$.
1. **Open ball** is $B(x,r) = \{y \in X; d(x,y) < r\}$.
2. **Closed ball** is $\overline{B}(x,r) = \{y \in X; d(x,y) \leq r\}$.
3. A is **open** if $\forall a \in A\ \exists q > 0 : B(a,q) \subset A$.
4. A is **closed** if its complement is open.

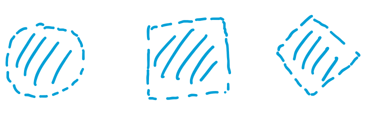


Examples: closed balls





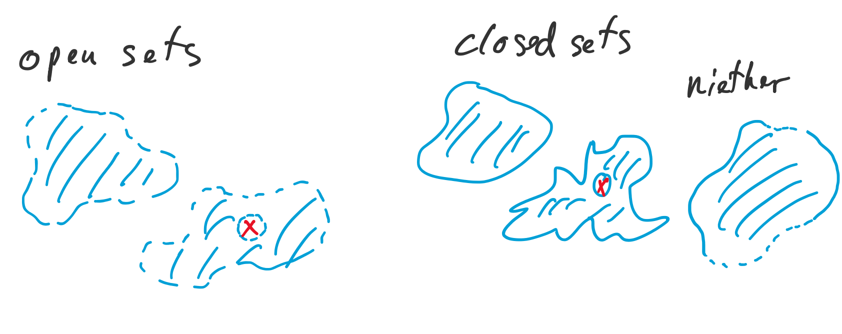
open balls



open sets

closed sets

niether

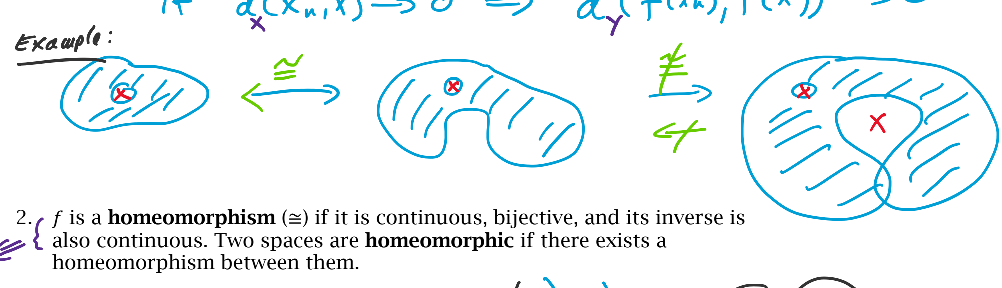


Def. Let $f: X \to Y$ be a map.
1. f is **continuous** if $\forall U \subset Y$ open, its preimage $f^{-1}(U) = \{y \in X: f(y) \in U\}$ is open.

↳ equivalently

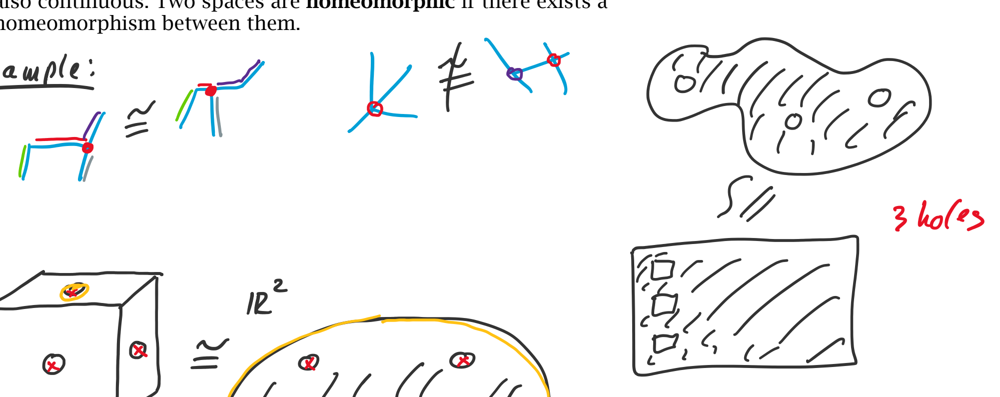
if $d_X(x_n, x) \xrightarrow{n\to\infty} 0 \Rightarrow d_Y(f(x_n), f(x)) \xrightarrow{n\to\infty} 0$

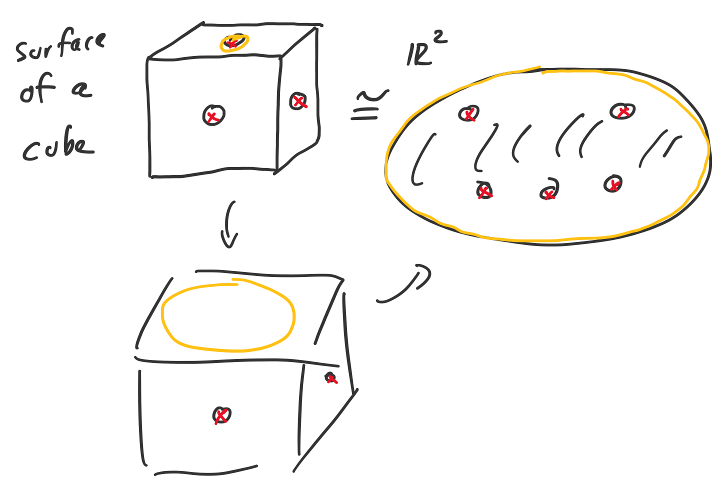
Example:



2. f is a **homeomorphism** (≅) if it is continuous, bijective, and its inverse is also continuous. Two spaces are **homeomorphic** if there exists a homeomorphism between them.

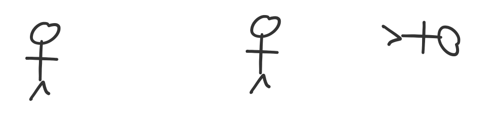
Example:



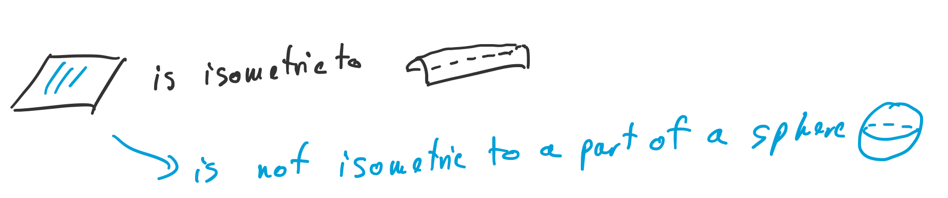


3. f is **isometry** if it preserves distances: $d(x,y) = d(f(x), f(y))$.

Example in the plane (with d_2). Isometries are translations, rotations and reflections.



Example with geodesic distance:

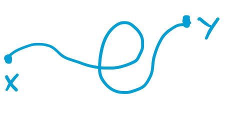


Notation: $n \in \mathbb{N}$

n-disc D^n (dimension) .... any space homeomorphic to the closed unit ball in $\mathbb{R}^n$ $\overline{B}_{\mathbb{R}^n}(a,1)$

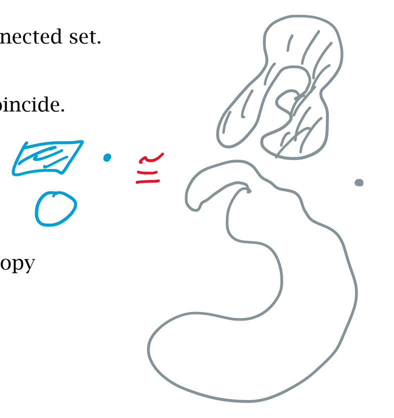
n-sphere S^n... any space homeomorphic to the boundary of $\overline{B}_{\mathbb{R}^{n+1}}(a,1)$.

Def. Let X be a metric space.
1. A **path** in X is a continuous map $[0,1] \to X$.
2. X is **path connected** if for each pair of points in it there exists a path between them.
3. X is **connected** if it can not be expressed as a union of two disjoint non-empty open sets.
4. A subset $A \subset X$ is a **(path) component** of X if it is a maximal (path) connected set.



For the spaces we will be considering the two concepts of connectedness coincide.

Theorem: Suppose $X \cong Y$. Then:
1. X is connected iff Y is
2. Both have the same number of components.



Def. Continuous maps $f, g: X \to Y$ are **homotopic** (≃), if there exists a homotopy between them:

continuous $H: X \times [0,1] \to Y$

$H|_{X \times \{0\}} = f$

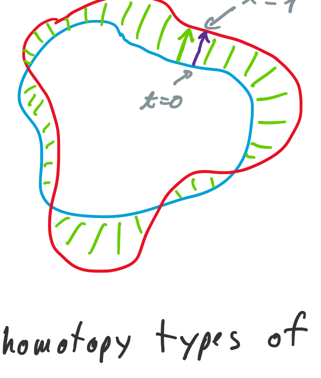
$H|_{X \times \{1\}} = g$.

Example: Any two maps into $\mathbb{R}^2$ (← Y) are homotopic

f

g

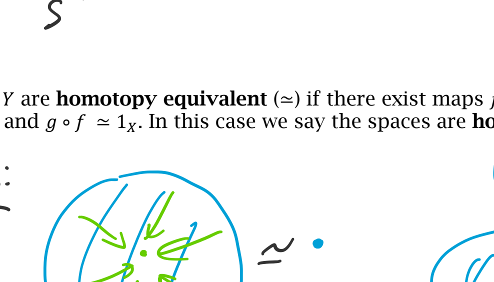
$X = S^1$



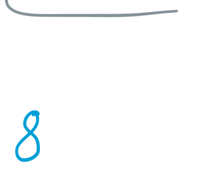
$H(x,t) = (1-t)f(x) + t \cdot g(x)$

Example: How many homotopy types of maps exist: 3

Example:

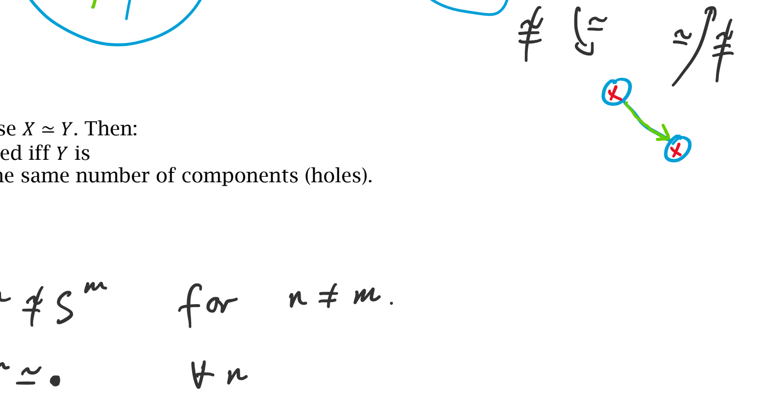


$f \not\simeq g$



Def. Spaces X, Y are **homotopy equivalent** (≃) if there exist maps $f: X \to Y, g: Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. In this case we say the spaces are **homotopy equivalent.**

Examples:

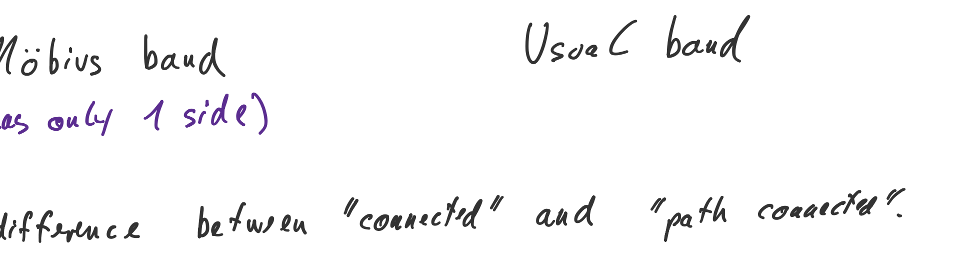


Theorem. Suppose $X \simeq Y$. Then:
1. X is connected iff Y is
2. Both have the same number of components (holes).

Examples:

$S^n \not\simeq S^m$ for $n \neq m$.

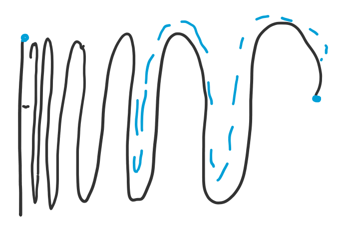
$D^n \simeq \bullet$ $\forall n$



Möbius band (has only 1 side)

Usual band

The difference between "connected" and "path connected".



is connected

NOT p. connected.