1. Find general solutions of linear differential equations below, then find the solution of the corresponding initial value problem.

   (a) $xy' + y = 3x^2 - 2x + 1$, $y(1) = 1$,
   (b) $y' + y = x + 1$, $y(0) = 2$,
   (c) $\sin(x)y' + y = 1$, $y(\pi/2) = 2$,
   (d) $y' + y/x = \cos(x)$, $y(\pi) = 0$,
   (e) $xy' + y = -\sin(x)$, $y(\pi) = 0$,
   (f) $y' - 2xy = 2x$, $y(1) = 0$.

2. **Orthogonal trajectories.** Let $f(x,y,c)$ be a function of three variables. View x and y as cartesian coordinates of a point in the plane, and view c as a parameter. The equation $f(x,y,c) = 0$ determines a family of curves in the plane $\mathbb{R}^2$. (For each fixed value of the parameter c we have a single level set of a function of two variables.) For such a family of curves we want to find its orthogonal trajectories – a new family of curves with the property that each curve in this family is orthogonal to each curve of the original family. This can be done by determining the differential equation satisfied by the family $f(x,y,c) = 0$ and replacing the derivative y' that appears in that differential equation by $-1/y'$.

   Find the orthogonal trajectories for the following families of curves. (You can leave the results in implicit form.)

   (a) $y = x^2 + a$ for $a \in \mathbb{R}$,
   (b) $y = \dfrac{a}{x}$ for $a \in \mathbb{R}$,
   (c) $y = ax^2$ for $a \in \mathbb{R}$,
   (d) $y = ax^n$ for $a \in \mathbb{R}$ (and $n \in \mathbb{N}$),
   (e) $(x+y)^2 = ax^2$ for $a \in \mathbb{R}$,
   (f) $x^2 + y^2 = r^2$ for $r \in \mathbb{R}$,
   (g) $(x-r)^2 + y^2 = r^2$ for $r \in \mathbb{R}$.

3. You are given the differential equation

   $$2xy - 9x^2 + (2y + x^2 + 1)y' = 0$$

   subject to the initial condition $y(0) = -3$.

   (a) Write down the functions $P(x,y)$ and $Q(x,y)$ such that the equation $P(x,y) + Q(x,y)y' = 0$ agrees with the equation above.
   (b) Verify that $P_y = Q_x$.
   (c) Find a function $f(x,y)$ such that $P = f_x$ and $Q = f_y$.
   (d) Solve the differential equation above.

4. Denoting $\mathbf{x} = [x,y]^{\mathsf{T}}$ find the general solutions to the system of differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ in case A is the following matrix:

   (a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

   Use Octave to draw phase portraits (solution trajectories for several initial values) for each of the systems above. How do eigenvalues of the matrix A affect the behaviour of the solutions?

5. The Van der Pol oscillator is a dynamical system with nonlinear damping satisfying this 2^{nd} order differential equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0.$$

   (a) Rewrite this 2^{nd} order d.e as a system of two first order d.e.'s. Plot the phase diagrams for $\mu = 1$ and a few chosen initial values.

   (b) Find the stationary points of this first order system and compute the eigenvalues of Jacobi matrix in these stationary points (use Octave). What can you deduce about the stability of the stationary points?

   (c) Find the initial value $[x(0), \dot{x}(0)]^{\mathsf{T}}$ for which $y(0) = \dot{x}(0) = 0$ which describes a periodical solution. (Start with $x(0) = x_0 > 0$ and $\dot{x}(0) = 0$ and then find x_1 for which the trajectory intersects the half line x $\geq 0, y = 0$. The determines a function $f\colon (0,\infty) \to (0,\infty)$, $x_0 \mapsto x_1$. You need to find a fixed point of this map, i.e. solve the equation $f(x) = x$.)