**Linear least squares method.** Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix with $m \geq n$. Let $\mathbf{b} \in \mathbb{R}^m$ be a vector. How would you find the orthogonal projection of $\mathbf{b}$ on to the column space of $A$, $C(A)$? (Assume that the columns of $A$ are linearly independent.)

1. We want to approximate a real function $f$ on the interval $[a, b]$ with a polynomial. We will do this (perhaps naïvely) by dividing the interval $[a, b]$ with $k+1$ equidistant points $a = x_0, x_1, \ldots, x_k = b$ and then find the coefficients of the polynomial $p(x)$ that is the best fit to the data in the table below according to the linear least squares method.

| $x_0$ | $x_1$ | $\cdots$ | $x_i$ | $\cdots$ | $x_k$ |
|---|---|---|---|---|---|
| $f(x_0)$ | $f(x_1)$ | $\cdots$ | $f(x_i)$ | $\cdots$ | $f(x_k)$ |

   (a) Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n$. Write the matrix $A$ of the corresponding linear system and the right-hand side of $\mathbf{b}$ according to the data in the table above.

   (b) Find the approximations of orders 0, 1 and 2 for the function $f(x) = \frac{x^2}{1+x^2}$ on the interval $[-1, 1]$ using the points $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.

   (c) Using `octave` approximate the function $g(x) = \frac{1}{1+25x^2}$ on $[-1, 1]$ with polynomials of order $0, 2, \ldots, 20$, dividing the interval $[-1, 1]$ with 21 equidistant points. Find the approximations for the exact data and for data with (artificially added) errors. Using the `plot` command plot the graphs of the original functions and all the approximations.

2. Use the linear least squares method to solve the following problem: In the plane $\mathbb{R}^2$ we have $n$ transmitters at known locations $(p_1, q_1), \ldots, (p_n, q_n)$. A receiver can measure the distances $d_1, \ldots, d_n$ from these transmitters. Given those distances, we would like to determine the position of the receiver. In the ideal case the measurements are exact and for each $i = 1, \ldots, n$ we have an equation

$$(x - p_i)^2 + (y - q_i)^2 = d_i^2.$$

   The solution of this system of equations then determines the unknown position of the receiver $(x, y)$.

   (a) The first problem is that the equations are *not* linear. But the difference of two consecutive equations is a linear equation. Write down these differences to obtain a system of $n-1$ linear equations.

   (b) Write the matrix $A \in \mathbb{R}^{(n-1)\times 2}$ of the system and the corresponding right-hand side $\mathbf{b} \in \mathbb{R}^{n-1}$. Additional problem is that the measurements are not exact which means that the system $A\mathbf{x} = \mathbf{b}$ (almost surely) has no solution.

   (c) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$. Write an `octave` function `X = sprejemnik([pi, qi], [di])` that finds the position of the receiver $X(x, y)$ given transmitter positions $(p_i, q_i)$ and distances $d_i$. (The positions $(p_i, q_i)$ are contained in an $n \times 2$ matrix and the distances $d_i$ are given in a column matrix of length $n$. The result $X$ should be a row vector `X = [x, y]`.)

(d) Test the function using artificial data from the files `oddajniki.txt` and `razdalje.txt` found on `Ucilnica`. You can import them into octave using the `load` command. Visualize the results.